A-369

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

February 17, 1958

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 1
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

The progress of the project is summarized below as follows:

1. The experimental equipment is essentially designed and construction is 75% complete. Schematics of the experimental set-up will be included in Progress Report No. 2.

2. The electronic equipment necessary for the determination of the pneumatic lag is on hand and is being readied for use on the project.

3. Theoretical work is being pursued and various approaches are being analyzed.

Future plans are as follows:

1. Within the next month it is anticipated that the experimental equipment will be complete and preliminary test data will be taken.

2. The analytical study will continue and it is anticipated that the preliminary experimental results will aid in pointing the way to the best approach for an analytical solution.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

Approved: ____________________

Thomas W. Jackson, Chief
Mechanical Sciences Division

ALD:O
cc: J. E. Fikes
T. G. Reed

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GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

May 26, 1958

Commander
U.S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 2
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

The progress of the project is summarized below as follows:

1. The experimental equipment is now in operation. A schematic of the experimental set-up is enclosed.

2. The electronic equipment has been installed and has been found to be capable of delivering the desired accuracy for time and pressure measurements.

3. Check-out runs have been made and the experimental program should be in progress by June 1, 1958.

4. Check runs on the IBM computer based on preliminary experimental data indicate that the theory in use at present gives the desired prediction of the lag inherent in the plumbing system.

Future plans are as follows:

1. It is hoped that the ascent phase can be completed during June 1958.

2. Further checks on the theory will be made as the experimental results become available.

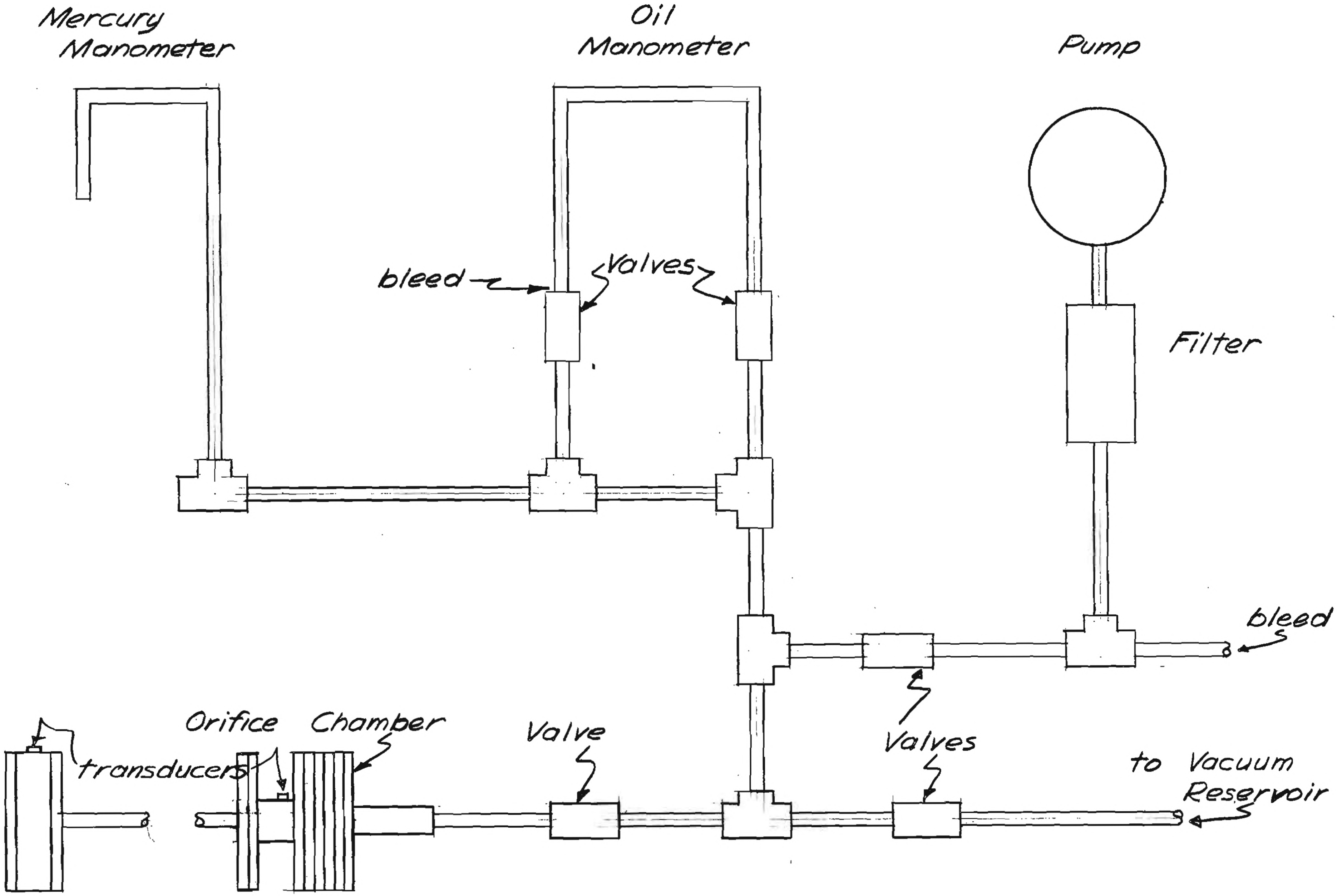
3. Work on the descent phase should be initiated in July 1958.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________________
Thomas W. Jackson, Chief
Mechanical Sciences Division

ALD:js
cc: J.E. Fikes
T.G. Reed



Pressure Lag Assembly - ABMA
No Scale

A-369

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION
ATLANTA, GEORGIA

July 2, 1958

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 3
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

Bi-Monthly Progress Report No. 2 covered, in addition to the period February and March, information which was applicable up to June 1, 1958. This report, therefore, will serve to establish the present status of the work which is as follows:

1. The equipment has been changed to measure pressures differentially in relation to a reference pressure instead of measuring differences by measuring absolute pressures and subtracting. This it is hoped will increase the accuracy of measurement.

2. Operation of the equipment and preliminary check out runs indicated the following problems:

   a. Drift in pressure readings will occur if care is not taken to consider storage volumes in parts of the system. This is due to the small orifices in the pressure gages.

   b. Leaks in the apparatus, especially around the heater leads to the orifices, have not been eliminated to date. A redesign of the orifice section may be necessary.

3. A new set of absolute pressure gages has been obtained at no cost to the contract.

Future work will consist of the following:

1. The new absolute pressure gages will be installed to permit more reliable and easier reading of pressures in the system.

2. Every effort will be made to eliminate the leaks which so far have made

the data taken inaccurate. At the present time various design modifications are under consideration. For instance, substituting quad-rings for O-rings around the orifices is being considered.

3. Every effort will be made to place the apparatus in operation by the end of July in order that the ascent runs can be initiated by August 1.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________________
Thomas W. Jackson, Chief
Mechanical Sciences Division

ALD:O

cc: J. E. Fikes
T. G. Reed

LIBRARY DOES NOT HAVE Bi-Monthly Progress
Report No. 4.

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GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA. GEORGIA

September 30, 1958

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: Bi-Monthly Progress Report No. 5
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems".

Gentlemen:

Bi-Monthly Progress Report No. 5 covers the period from August 1, 1958 to September 30, 1958.

1. The measurements of the static pressure lag for the ascent phase of the program have been reduced, plotted, and analyzed. The results indicate that some repeat runs must be made because of difficulties encountered with the electronic counter circuit.

2. The descent runs have been made and are being reduced at present.

3. Initial runs on an IBM 740 computer have been made to correlate the theory with experiment. Further runs on the computer will be made shortly.

Future work will consist of the following:

1. Reruns of part of the ascent data.

2. Reduction and plotting of the descent data.

3. Further work on the correlation of theory and experiment using the IBM 704 computer.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________

Thomas W. Jackson, Chief
Mechanical Sciences Division

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GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

March 3, 1959

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 6
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

Bi-Monthly Progress Report No. 6 covers the period from October 1, 1958 to November 30, 1958.

1. The electronic timer has been rebuilt and the ascent phase of the program has been rerun.

2. The ascent data has been reduced and plotted.

3. Theoretical calculations correlate with the ascent experimental data.

Future work will consist of the following:

1. By mutual agreement with ABMA personnel at Huntsville, Alabama new equipment will be designed in order that continuous type input trajectories can be applied to the system.

2. The descent phase will be rerun with the modified equipment.

3. Further work on the correlation of theory and experiment will be carried on.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________________
Thomas W. Jackson, Chief
Mechanical Sciences Division

cc: J. E. Fikes
T. G. Reed

ALD:pll

A-369

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

March 3, 1959

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 7
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

Bi-Monthly Progress Report No. 7 covers the period from December 1, 1958 to January 31, 1959.

1. The equipment for producing continuous type input trajectories has been designed and built.

2. Preliminary runs indicate that the desired trajectories can be realized within reasonable limits.

3. Correlation of theory and experiment using preliminary data appears very encouraging.

Future work will consist of the following:

1. The descent phase of the program will be run using five independent trajectories, 3 tube inside diameters, and 4 tube lengths.

2. The descent data will be reduced and plotted.

3. Correlation of theory and experiment will be undertaken using the theory developed during the course of this investigation.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________________
Thomas W. Jackson, Chief
Mechanical Sciences Division

cc: J. E. Fikes
T. G. Reed

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GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

April 22, 1959

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 8
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

Bi-Monthly Progress Report No. 8 covers the period from February 1, 1959 to March 31, 1959.

1. The descent phase has been run using 5 trajectories, 4 inside tube diameters, 4 tube lengths, and 5 receiver volumes.

2. The experimental data for the descent phase is being reduced at present.

3. The degree of correlation between theory and experiment for the descent phase is being determined.

Future work will consist of the following:

1. The experimental apparatus will be modified to accomodate the running of the ascent phase.

2. The reduction of the descent data will be completed.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________________
Thomas W. Jackson, Chief
Mechanical Sciences Division

cc: J. E. Fikes
T. G. Reed

A-369

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

June 10, 1959

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 9
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

Bi-Monthly Progress Report No. 9 covers the period from April 1, 1959 to May 31, 1959.

1. The ascent phase has been run using various values of line length, diameter, and receiver volume.

2. The experimental data for the ascent and descent phases have been reduced and are being correlated with theory.

Future work will consist of the following:

1. Correlation of experiment with theory.

2. Writing of final report.

Respectfully submitted,

Arnold L. Ducoffe
Project Director

APPROVED: ______________________
Thomas W. Jackson, Chief
Mechanical Sciences Division

cc: J. E. Fikes
T. G. Reed

A-369

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

July 20, 1959

Commander
U. S. Army Ordnance Corps
Army Ballistic Missile Agency
Huntsville, Alabama

Attention: ORDAB-HSI

Subject: Bi-Monthly Progress Report No. 9 (Addendum)
Contract No. DA-01-009-ORD-595
"A Theoretical and Experimental Study for the Prediction of Pneumatic Pressure Lag Inherent in Typical Ballistic Missile Plumbing Systems"

Gentlemen:

The addendum to Bi-Monthly Progress Report No. 9 contains the following information.

1. Development of the theoretical equation of motion which is being employed to correlate theory and experiment.

2. Some typical graphs showing the correlation between theory and experiment for the impulse-type trajectories (ascent) as well as for the continuous type trajectories (ascent and descent).

Respectfully submitted,

Arnold L. Ducoffe,
Project Director

APPROVED:

Thomas W. Jackson, Chief
Mechanical Sciences Division

cc: J. E. Fikes
T. G. Reed

ALD:pll

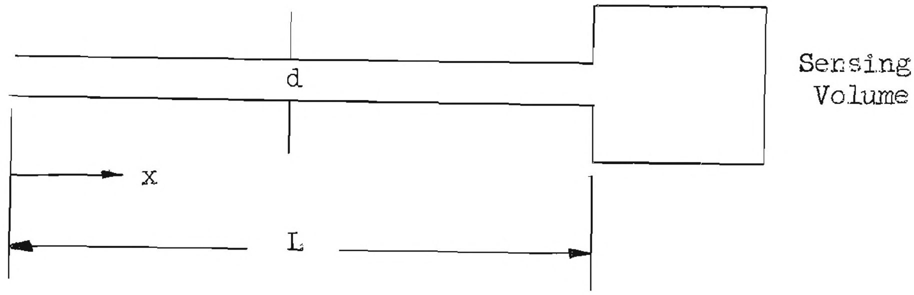
THEORY

The derivation of the response pressure equation is based upon the assumption of continuum flow, tubing of constant cross-sectional area, and a fully developed laminar flow over the entire length of tubing. Changes of state of the fluid medium are assumed to take place according to the isothermal law; it being felt that this assumption is justified on the grounds that the large ratio of internal surface area of the system to internal volume of the system, together with the large ratio of mass of metal in the system to mass of air in the system, make the instrument itself a relatively large capacity heat reservoir.

A quasi-steady solution is developed, based on the supposition that the mass flow is independent of spatial position within the system and dependent only upon time. The Hagen-Poiseuille Law for steady, incompressible, fully developed flow, with compressibility introduced through the isothermal relation, is used to determine the mass flow. Finally, the isothermal relation is used to relate the rate of change of mass in the system to the rate of change of response pressure. The equation resulting from the subsequent elimination of mass flow is a non-linear, first order differential equation for response pressure. Several solutions of this equation, with an empirically determined constant being substituted for that predicted by the theory, were obtained by the method of isoclines, and comparison is made with the experimental data. An empirical relation is presented for predicting the value of this constant.

DERIVATION OF EQUATION

If m' and Q are respectively the mass flow and volumetric flow past any point in the system shown below



then the Hagen-Poiseuille Law for quasi-steady, incompressible, fully developed flow states

$$m' = \rho Q = - \frac{\pi d^4}{128\mu} \rho \frac{\partial p}{\partial x} \qquad (1)$$

where

ρ is mass density

μ is coefficient of viscosity

p is pressure

d is diameter of tubing

x is spatial coordinate, measured along axis of tube

Introducing compressibility through the isothermal relation $p = k\rho$, where $k = RT$, R being the gas constant and T the absolute temperature, equation (1) is modified and gives,

$$m' = \rho Q = - \frac{\pi d^4}{128\mu k} p \frac{\partial p}{\partial x} = - \frac{\pi d^4}{128\mu k} \frac{\partial}{\partial x}\left(\frac{p^2}{2}\right) \qquad (2)$$

Now the equation of continuity for a compressible flow of the quasi-steady nature supposed may be written

$$\frac{\partial}{\partial x}(\rho Q) = 0 \tag{3}$$

or, differentiating equation (2), $\frac{\partial^2}{\partial x^2}(p^2) = 0$ which can be integrated immediately to yield $p^2 = Ax + B$, where A and B are functions of time. (4)
The boundary conditions are

$$\begin{aligned} &(a) \quad \text{at } x = 0,\ p = p_i \\ &(b) \quad \text{at } x = L,\ p = p_r \end{aligned} \tag{5}$$

Insertion of the boundary conditions (5) into equation (4) gives

$$p^2 = (p_r^2 - p_i^2)\frac{x}{L} + p_i^2 \tag{6}$$

Substituting this result into equation (2), we obtain

$$m' = -\frac{\pi d^4}{256\mu kL}(p_r^2 - p_i^2) \tag{7}$$

Also, the equation of state $p = \rho RT$ may be written

$$\bar{p}\,V = mRT \tag{8}$$

where $$\bar{p} = \frac{1}{V}\int_V p\,dv \tag{9}$$

V being the volume of the system, so that differentiation of equation (8) gives, for an isothermal process

$$\frac{dm}{dt} = m' = \frac{V}{RT}\frac{d\bar{p}}{dt} \tag{10}$$

However, the pressure distribution is parabolic with length (equation 6), and for the experiments run the volume of the tubing is always smaller than the sensing volume. Both of these latter effects tend to weight the mean pressure defined by equation (9) towards p_r, thus it is felt that little error is introduced by writing equation (10) as

$$\frac{dm}{dt} = m' = \frac{V}{RT}\,\frac{dp_r}{dt} = \frac{V}{k}\,\frac{dp_r}{dt} \tag{11}$$

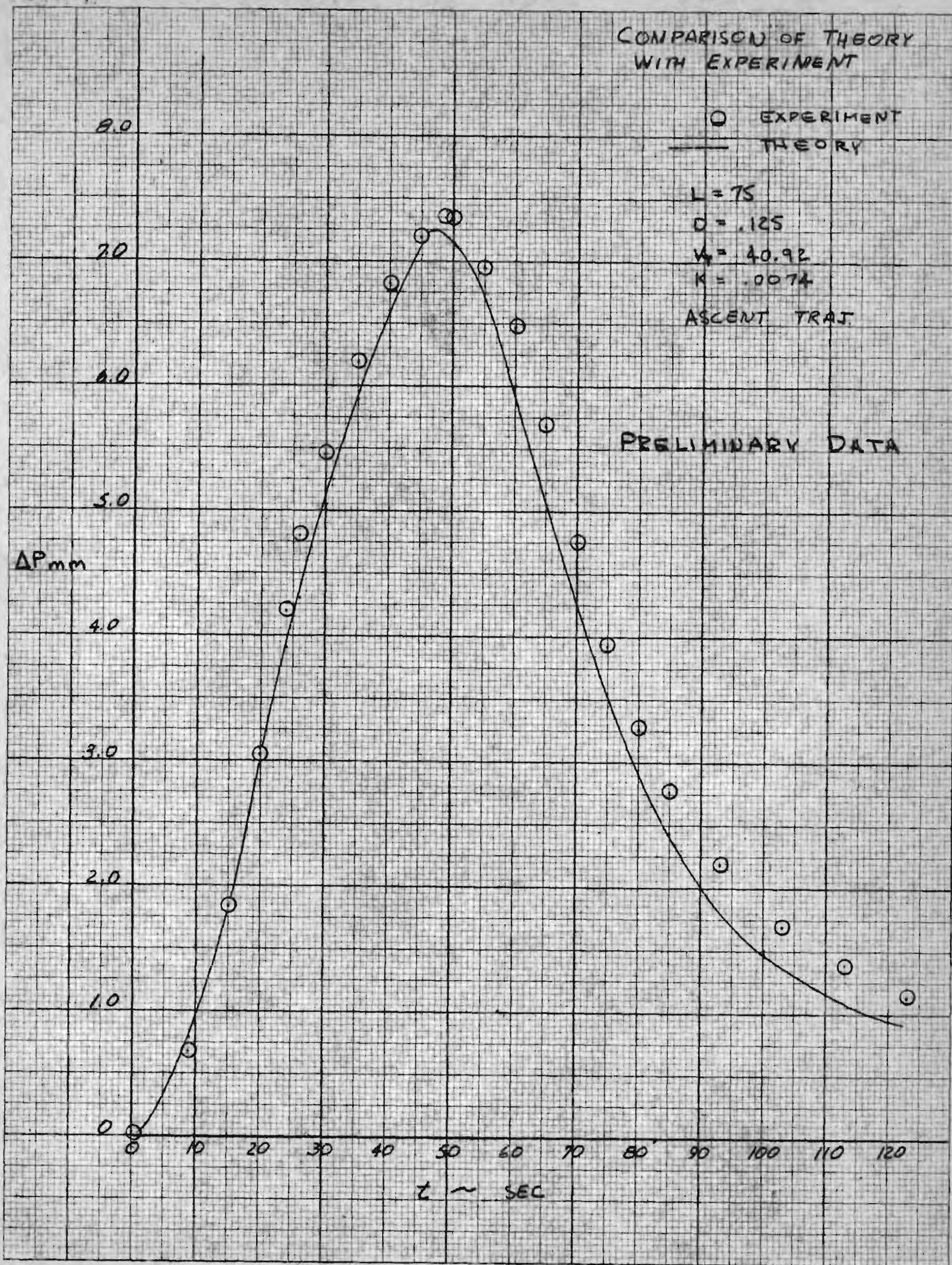
Finally, equating the expressions for m' contained in equations (7) and (11), we obtain

$$\frac{dp_r}{dt} = -\frac{\pi d^4}{256\mu VL}(p_r^2 - p_i^2) \tag{12}$$

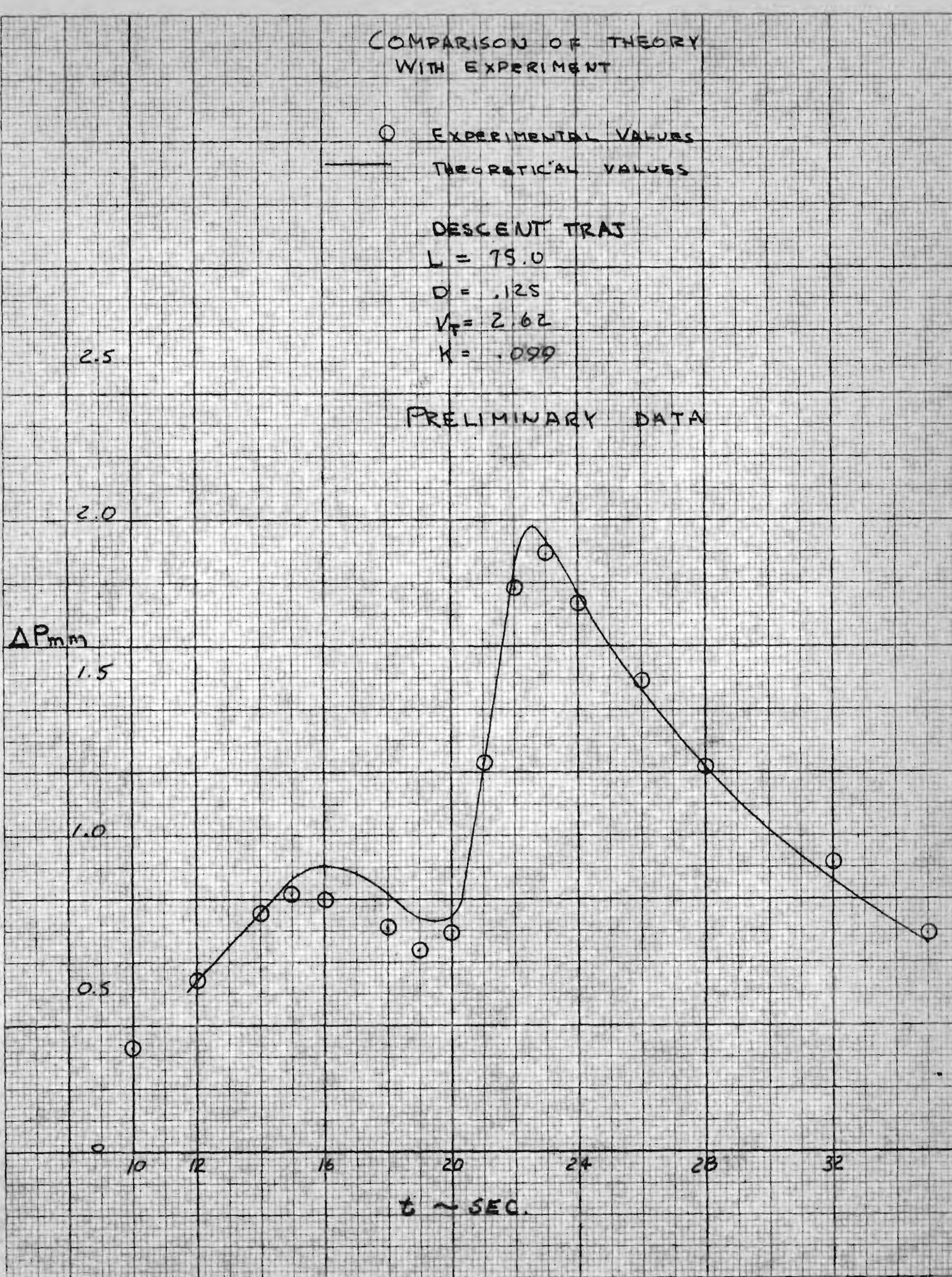
or, putting $K = \dfrac{\pi d^4}{256\mu VL}$ (13)

$$\frac{dp_r}{dt} = K(p_i^2 - p_r^2) \tag{14}$$

which is the equation for response pressure for quasi-steady, compressible viscous flow.



COMPARISON OF THEORY WITH EXPERIMENT
EXPERIMENT
THEORY
L = 75
D = .125
V = 40.92
K = .0074
ASCENT TRAJ.
PRELIMINARY DATA
ΔP mm
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
0
10
20
30
40
50
60
70
80
90
100
110
120
t ~ SEC



COMPARISON OF THEORY
WITH EXPERIMENT
O EXPERIMENTAL VALUES
THEORETICAL VALUES
DESCENT TRAJ
L = 75.0
D = .125
V_T = 2.62
K = .099
PRELIMINARY DATA
ΔPmm
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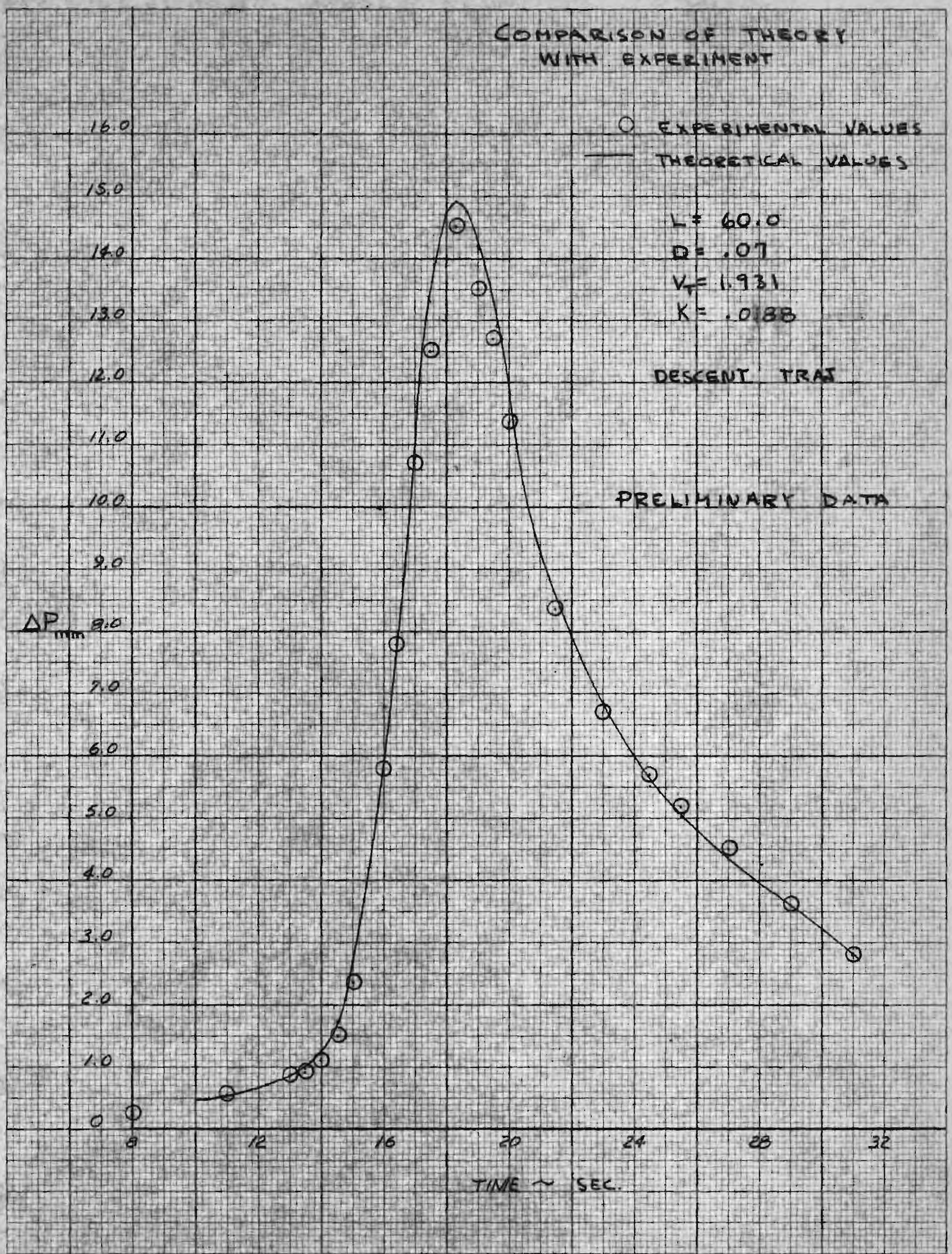
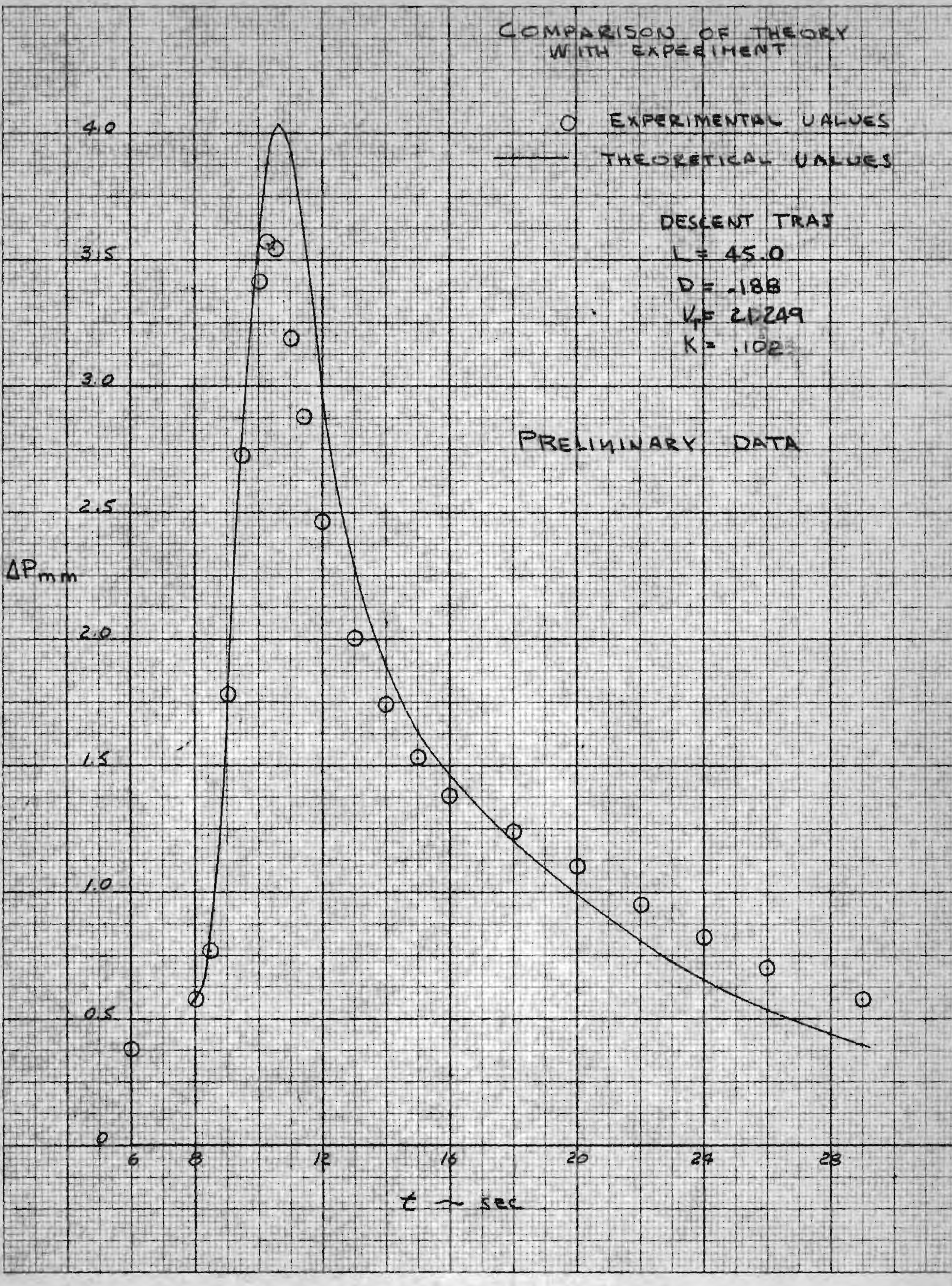
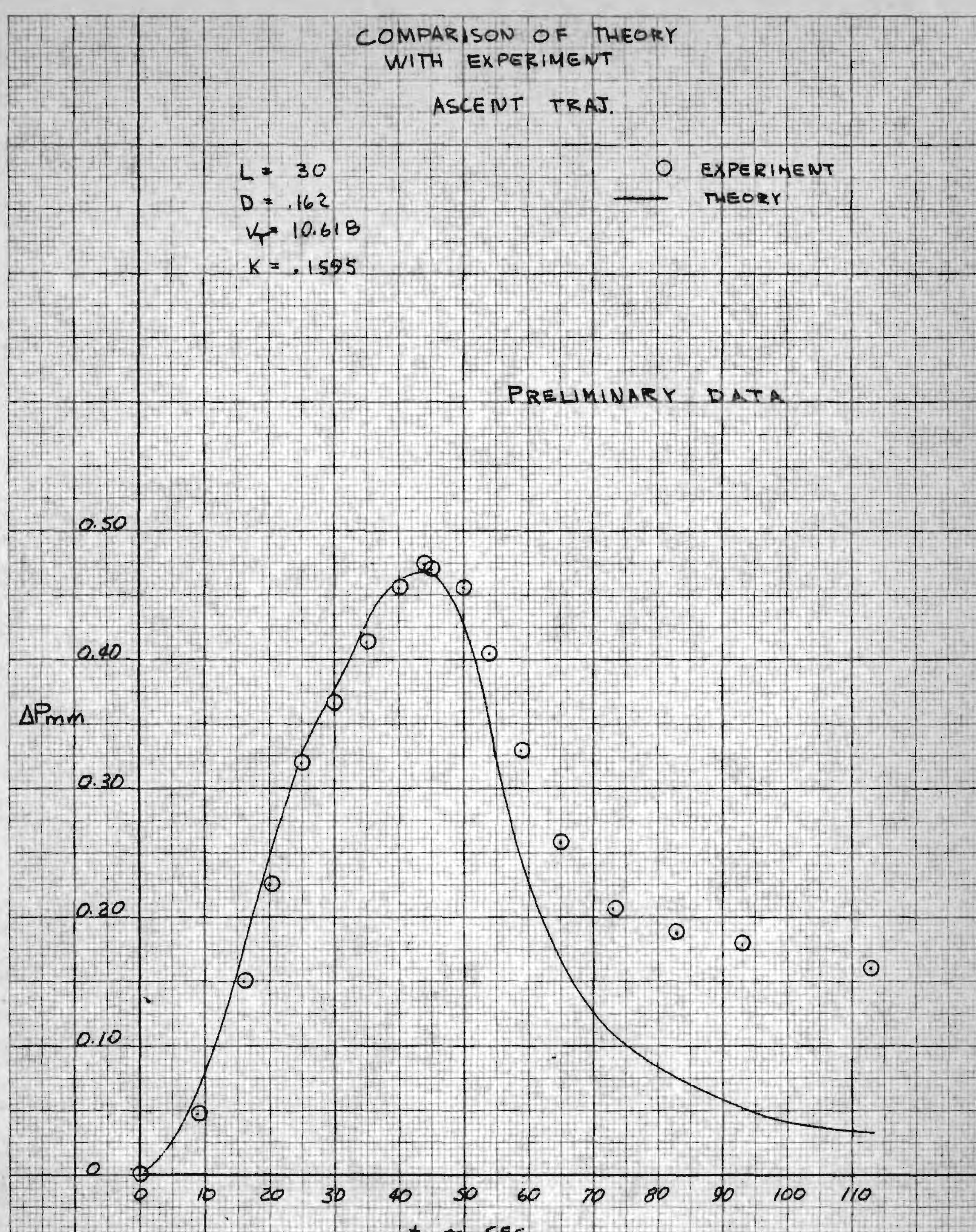
1.5
1.0
0.5
0
10
12
16
20
24
28
32
t ~ SEC.


COMPARISON OF THEORY WITH EXPERIMENT
EXPERIMENTAL VALUES
THEORETICAL VALUES
L = 60.0
D = .07
V_T = 1.931
K = .0188
DESCENT TRAJ
PRELIMINARY DATA
ΔP mm
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
11.0
12.0
13.0
14.0
15.0
16.0
8
12
16
20
24
28
32
TIME ~ SEC.


COMPARISON OF THEORY WITH EXPERIMENT
EXPERIMENTAL VALUES
THEORETICAL VALUES
DESCENT TRAJ
L = 45.0
D = .188
V_1 = 21249
K = .102
PRELIMINARY DATA
ΔP mm
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
6
8
12
16
20
24
28
t ~ sec


COMPARISON OF THEORY
WITH EXPERIMENT
ASCENT TRAJ.
L = 30
D = .162
V_T = 10.618
K = .1595
EXPERIMENT
THEORY
PRELIMINARY DATA
0.50
0.40
ΔPmm
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0
10
20
30
40
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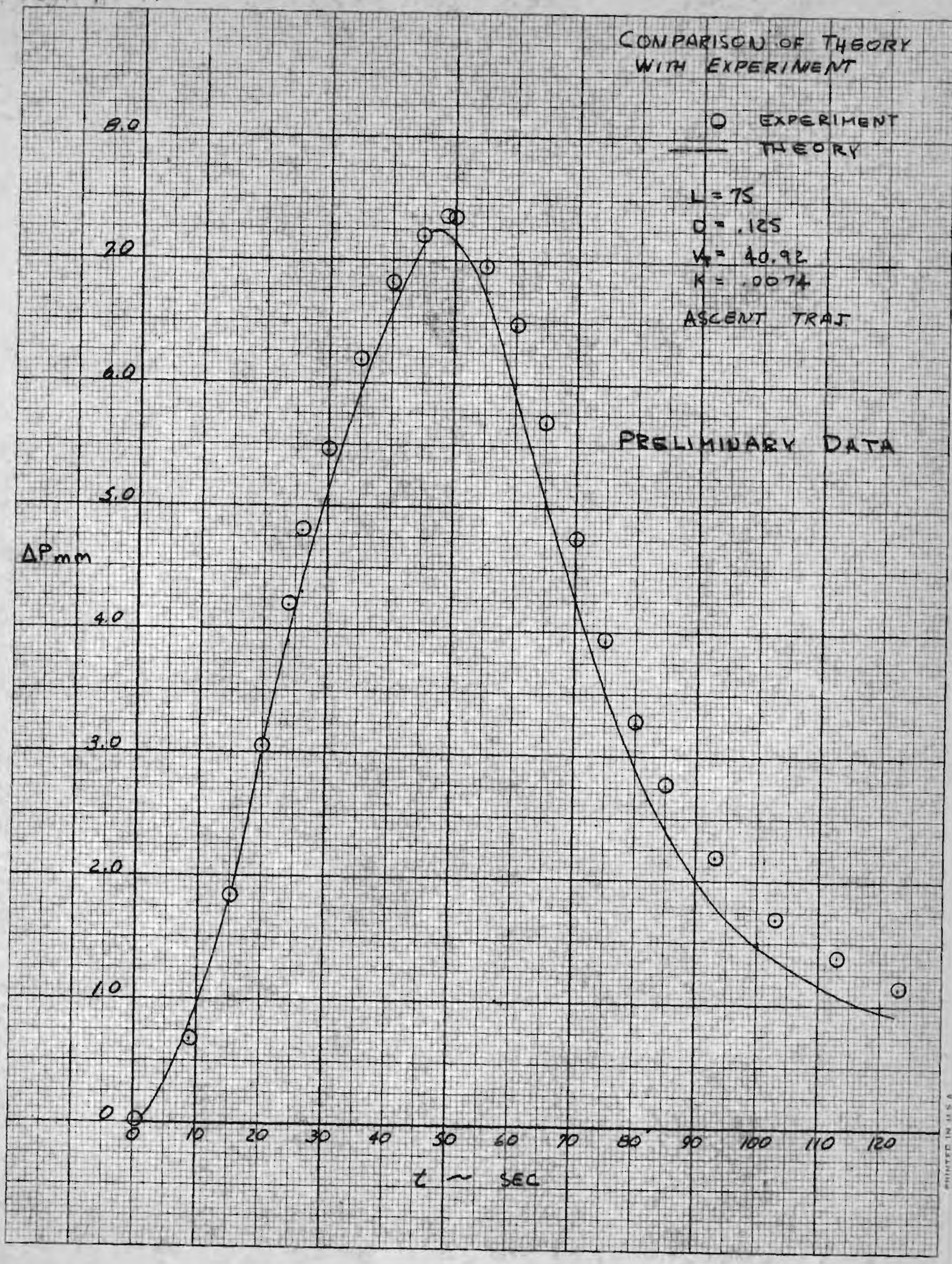
100
110
t ~ SEC



COMPARISON OF THEORY
WITH EXPERIMENT
EXPERIMENT
THEORY
L = 75
D = .125
V = 40.92
K = .0074
ASCENT TRAJ
PRELIMINARY DATA
ΔPmm
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
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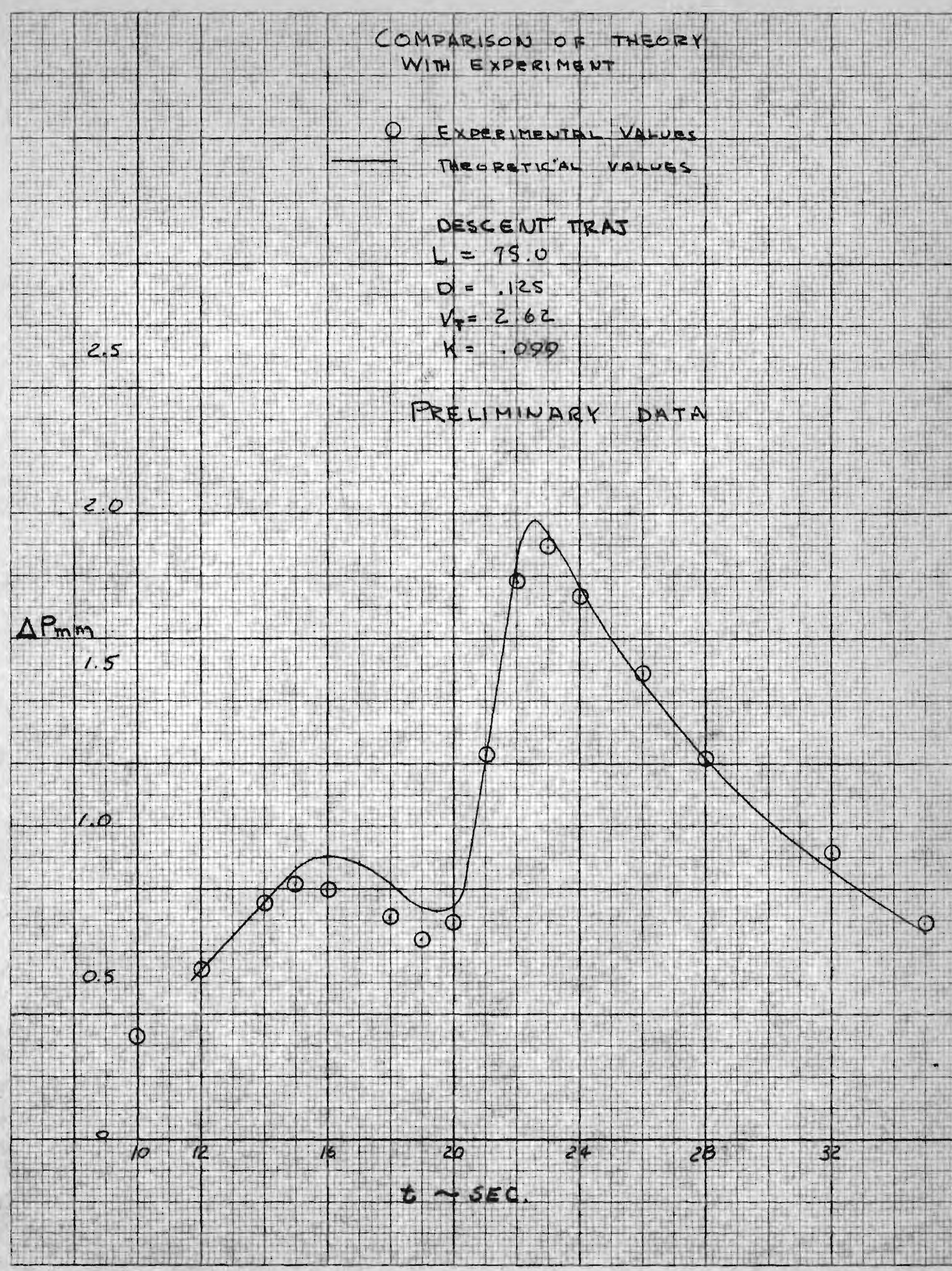
t ~ SEC



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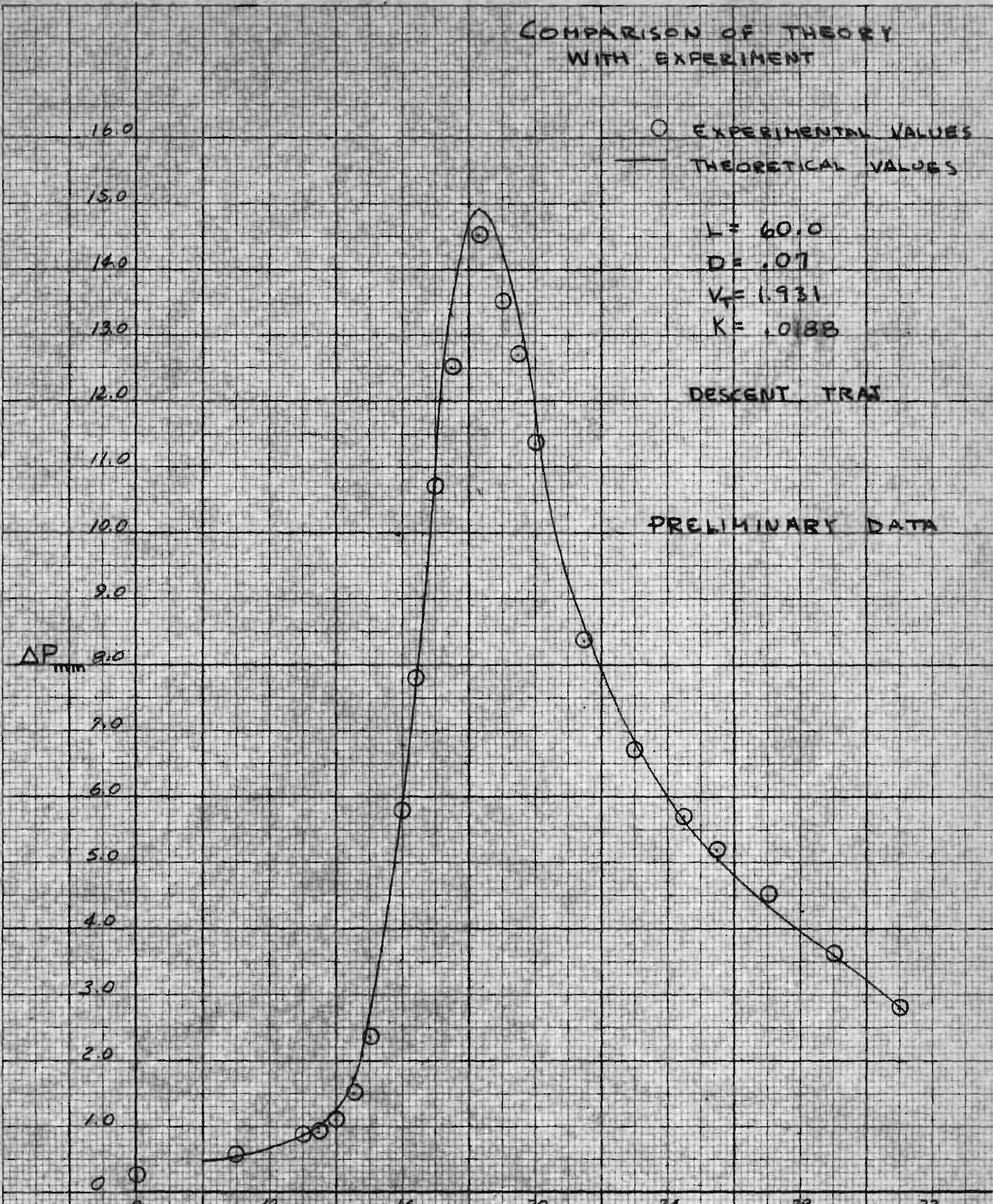

COMPARISON OF THEORY WITH EXPERIMENT
○ EXPERIMENTAL VALUES
— THEORETICAL VALUES
L = 60.0
D = .07
V_T = 1.931
K = .0188
DESCENT TRAJ
PRELIMINARY DATA
ΔP_{mm}
16.0
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
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16
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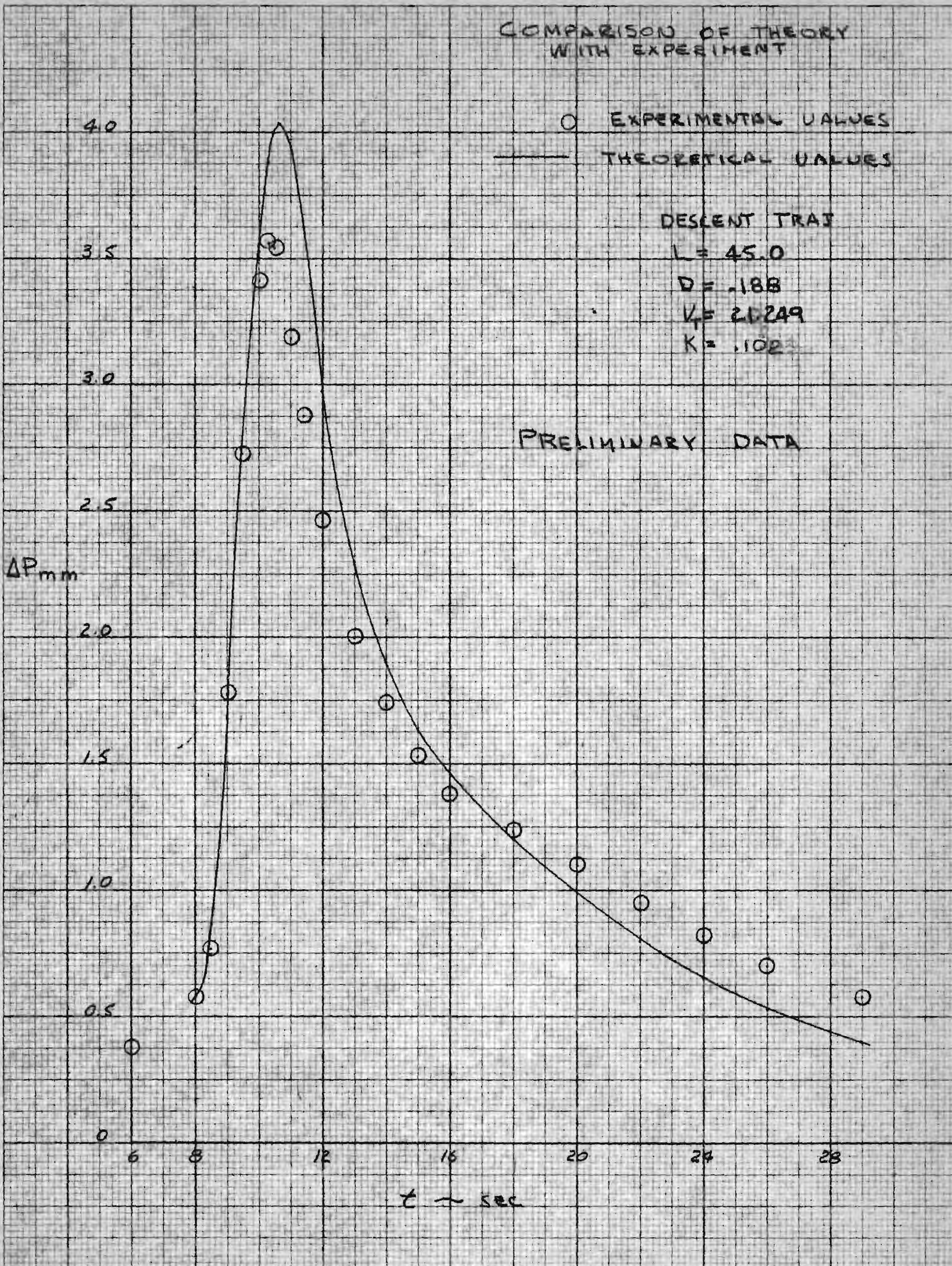
24
28
32
TIME ~ SEC.


COMPARISON OF THEORY
WITH EXPERIMENT
○ EXPERIMENTAL VALUES
—— THEORETICAL VALUES
DESCENT TRAJ
L = 45.0
D = .188
V_T = 21.249
K = .102
PRELIMINARY DATA
ΔP_{mm}
4.0
3.5
3.0
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2.0
1.5
1.0
0.5
0
6
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12
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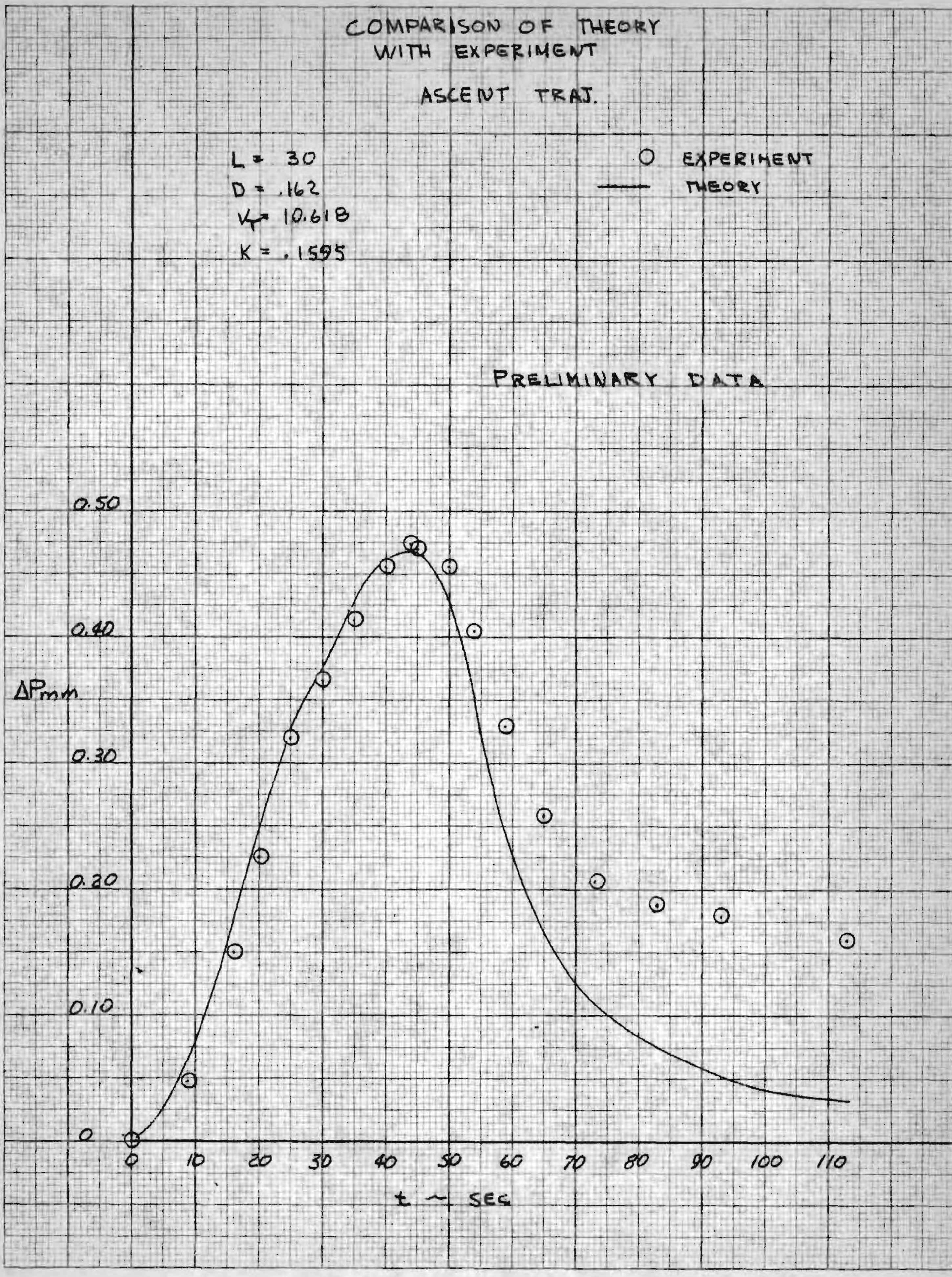
20
24
28
t ~ sec


COMPARISON OF THEORY
WITH EXPERIMENT
ASCENT TRAJ.
L = 30
D = .162
V_T = 10.618
K = .1555
EXPERIMENT
THEORY
PRELIMINARY DATA
ΔPmm
0.50
0.40
0.30
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0.10
0
0
10
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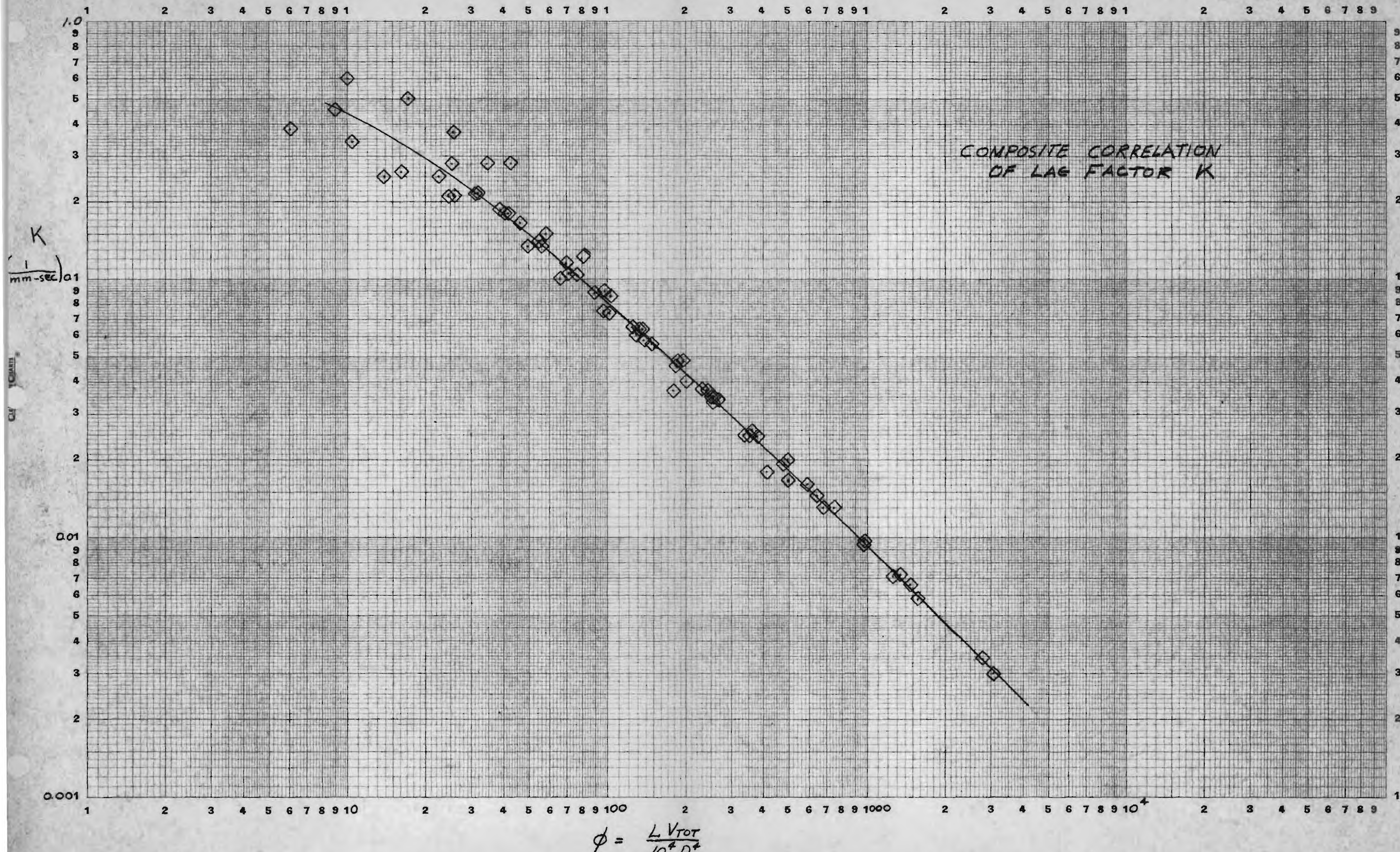
30
40
50
60
70
80
90
100
110
t ~ SEC


COMPOSITE CORRELATION OF LAG FACTOR K
K
$\left(\frac{1}{mm\text{-}sec}\right)$
1.0
0.1
0.01
0.001
1
10
100
1000
10^4
$\phi = \frac{L\,V_{TOT}}{10^4 D^4}$

FINAL REPORT

PROJECT A-369

A THEORETICAL AND EXPERIMENTAL STUDY FOR THE PREDICTION OF PNEUMATIC PRESSURE LAG INHERENT IN TYPICAL BALLISTIC MISSILE PLUMBING SYSTEMS

CONTRACT WITH THE
ARMY BALLISTIC MISSILE AGENCY
DA-01-009-ORD-595

1 DECEMBER 1957 THROUGH 30 JUNE 1959

Approved by:

Thomas W. Jackson
Chief, Mechanical Services Division

Released by:

for J. E. Boyd
Director

Submitted by:

Arnold L. Ducoffe
Project Director

Frank M. White, Jr.
Project Engineer

ACKNOWLEDGMENTS

The authors wish to extend their thanks to the following graduate students who participated in the performance of this work: K. Ball, J. Cremin, R. Hiers, B. Kowalsky, D. Pirie. In addition the help of Messrs Cook, Slocum and Van Tassel, who helped fabricate the equipment and also aided in the experimental phases, is greatly appreciated. Thanks are also extended to Miss B. Daniels for her help in the reduction of the data and to Miss P. Lane for her aid in the preparation of this report.

TABLE OF CONTENTS

LIST OF FIGURES

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SUMMARY

This report presents the results of a theoretical and experimental study of the pneumatic lag inherent in typical missile plumbing systems when subjected to impulse and continuous-type transient input pressure functions during ascending and descending flight.

Experimental tests were run for various system geometries using line length, line diameter, and sensing volume as variables. In addition, trajectories were simulated by applying various transient input pressure functions to one end of the tubing and measuring the pressure difference between the input and response end of the tubing.

An analytical study which assumes a quasi-steady, fully-developed, laminar, tube flow with isothermal changes of state is presented. The equation of motion for the flow is non-linear and solutions are obtained using step-by-step integrations or the method of isoclines.

An empirically determined system parameter which is shown to be a function of system geometry and independent of the trajectory is determined and the resulting correlation of theory with experiment is quantitative.

The results of this analysis are limited to a simple series system having a line and sensing volume. Further investigation of the effects of inlet temperature, connector fittings having inside diameters smaller than the tube diameter, and the effect of breathing(inflow and outflow at the surface port) is considered necessary for the complete solution of the use of plumbing systems for the measurement of pressure.

LIST OF SYMBOLS

A ------- Constant

B ------- Constant

C ------- Time dependent variable

C_1 ------ Time dependent variable

d ------- Inside diameter of tubing

h ------- Time increment

k ------- Constant (= RT)

K ------- System parameter

L ------- Line length

L/d ----- Tubing length to diameter ratio

m ------- Mass of air in system

m' ------ Rate of mass flow

P(t) ---- Transient input pressure

P_1 ------ Transient response pressure

p ------- Pressure

$\bar{p}$ ------- Average pressure in system

p_i ------- Transient input pressure

p_r ------ Transient response pressure

$p_r(t)$ --- Transient response pressure

$p_r(t+h)$ - Transient response pressure at time = (t+h)

p_{r_o} ----- Response pressure at time zero

p'_r ----- Derivative of transient response pressure $(\frac{dp_r}{dt})$

$p'_r(t)$ -- Derivative of transient response pressure $(\frac{dp_r}{dt})$

Δp ---- Pressure difference or pressure lag

$\Delta p(t)$ - Transient pressure difference or pressure lag

Q ------- Volume flow

R	Gas constant
T	Absolute temperature
t	Time
V	Total volume of air in system
V_s	Volume of air in sensing or response part of system
V_t	Volume of air in tubing
x	Spatial coordinate
λ	Time lag constant
μ	Coefficient of viscosity
π	Constant($= 3.1416$)
ρ	Mass density of air
ϕ	Dimensionless geometric parameter($= \frac{LV}{10^4 d^4}$)

INTRODUCTION

The measurement of transient-input pneumatic pressures by means of a series system comprised of a length of constant area tubing attached to a sensing volume has been the subject of numerous investigations during the past twenty-five years. The early attempts[1,2,3*] to predict pneumatic lag were concerned primarily with the measurement of pitot tube static pressures for aircraft whose speeds lay in the 200-400 mph category. In this velocity regime the transients encountered during maneuvering or diving flight were small in nature such that the pneumatic lag could be predicted to a reasonable degree of accuracy by a mathematical model in the form of a first order linear differential equation, namely,

$$\lambda \frac{dP_1}{dt} + P_1 = P(t) \qquad (1)$$

where λ is a lag time constant, P_1 is the pressure in the sensing volume, $P(t)$ is the input transient pressure function, and t is the time. Equation (1) was used extensively prior to and immediately after World War II. However, the introduction of transonic and supersonic aircraft, high speed missiles, etc., following World War II resulted in transient pressure inputs which could not be represented mathematically by Equation(1). During the late forties and early fifties Vaughn[4] recognized the need for a new analytical approach to the pneumatic lag problem. Vaughn's work consisted of the development of a non linear theory based upon the mass flow through a sharp-edged orifice[5] attached

* Numbered superscripts refer to the references located in the bibliography.

to a sensing volume. The resulting mathematical expression is given as

$$\frac{dP_1}{dt} = \frac{T_1}{T} B \sqrt{P^2 - P_1^2} \qquad (2)$$

where P_1 is the sensing volume pressure, P is the transient input pressure, T_1 is the absolute temperature in the sensing volume, T is the absolute input temperature, and B is a function of the plumbing geometry and input temperature. Vaughn's equation appears to give satisfactory results for systems with short line lengths subjected to small step inputs. However, for systems comprised of long lines and subjected to high-rate transient inputs, the use of Equation (2) results in doubtful correlation of theory with experiment.

On the basis of the apparent lack of information on the subject of pneumatic lag for typical missile plumbing systems subjected to high-rate transient inputs, the present analysis was undertaken. The work reported herein will consist of an analytical and experimental investigation for the prediction of pneumatic pressure lag encountered by typical missile plumbing systems. The input pressure functions will consist of (a) impulse functions such as encountered by multi-stage missiles during ascending flight, and (b) continuous type functions such as those imposed under the conditions of the ascending and descending flight of a single stage missile. In addition the results will be applicable to missiles or bombs dropped from high altitude.

THEORY

The derivation of the equation of motion for the flow of air in typical missile plumbing comprised of a length of constant area tubing attached to a sensing volume, Figure 1, is based upon the assumption of a continuum flow. In

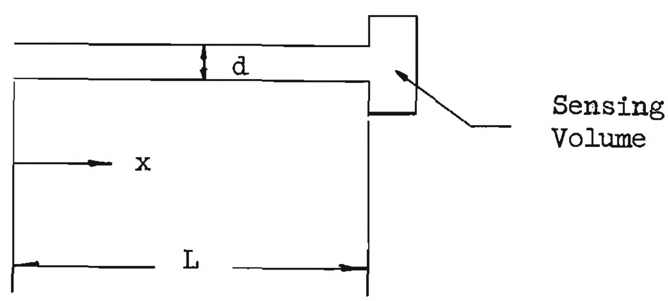


Figure 1.

addition it is also assumed that a fully-developed laminar flow exists over the entire tubing length. Starting with the Hagen-Poiseuille equation[5] for the steady, incompressible, fully-developed, laminar flow in a tube, the rate of mass flow is given by

$$m' = \rho\, Q = \frac{-\pi d^4}{128\mu} \rho \frac{\partial p}{\partial x} \qquad (3)$$

where

m'	is the rate of mass flow,
ρ	is the mass density,
Q	is the volumetric rate of flow,
d	is the inside diameter of the tube,
μ	is the coefficient of viscosity,
p	is the pressure, and
x	is the spatial coordinate, measured along the tube axis.

The effect of compressibility is introduced through the isothermal relation $p = k\rho$, where $k = RT$, R being the gas constant and T the absolute temperature. The assumption of isothermal changes of state is based upon the following argument. The ratio of internal surface area of the tubing to the internal volume of the system is large and the ratio of the mass of metal of the system to the mass of air in the system is also large; thus the metallic system serves as a heat source or sink to the air flow and tends to maintain the air temperature essentially constant. Introducing the isothermal assumption into equation (3) results in

$$m' = \rho Q = \frac{-\pi d^4}{128\mu k} p \frac{\partial p}{\partial x} \tag{4}$$

Equation (4) is rewritten as

$$m' = \rho Q = \frac{-\pi d^4}{128\mu k} \frac{\partial}{\partial x}\left(\frac{p^2}{2}\right) \tag{5}$$

The concept of quasi-steady motion is applicable to Equation (5) if the assumption is made that the rate of mass flow through the tubing is independent of spatial position and dependent upon time only. The quasi-steady motion is introduced through the continuity equation which is written as

$$\frac{\partial}{\partial x}(\rho Q) = 0 \tag{6}$$

Differentiating Equation (5) with respect to x and substituting Equation (6) into the result, gives

$$\frac{\partial^2}{\partial x^2}(p^2) = 0 \tag{7}$$

Equation (7) can be integrated (with time held constant) and yields

$$p^2 = Cx + C_1 \tag{8}$$

where C and C_1 are functions of time.
The boundary conditions applicable to equation (8) are,

$$\begin{aligned} &a)\ \text{at}\ x = 0,\ p = p_i \\ &b)\ \text{at}\ x = L,\ p = p_r \end{aligned} \tag{9}$$

Substituting the boundary conditions, (equation (9)), into equation (8) results in

$$p^2 = (p_r^2 - p_i^2)\frac{x}{L} + p_i^2 \tag{10}$$

Substitution of equation (10) into equation (5) gives

$$m' = \frac{-\pi d^4}{256 \mu kL}(p_r^2 - p_i^2) \tag{11}$$

The equation of state may be written as

$$\bar{p}\, V = mRT \tag{12}$$

where

$$\bar{p} = \frac{1}{V} \int_V p dV = \text{mean pressure in the system} \tag{13}$$

V = volume of the system

m = mass of air in the system

Assuming as before that the change of state takes place isothermally, differentiation of equation (12) yields

$$\frac{dm}{dt} = m' = \frac{V}{RT}\,\frac{d\bar{p}}{dt} \tag{14}$$

Equations (11) and (14) give expressions for the rate of mass flow through the system. Equating equations (11) and (14) gives

$$\frac{V}{RT}\,\frac{d\bar{p}}{dt} = \frac{-\pi\, d^4}{256\mu kL}\,(p_r^2 - p_i^2) \tag{15}$$

Substituting $k = RT$ on the left side of equation (15) and simplifying, gives

$$\frac{d\bar{p}}{dt} = \frac{-\pi\, d^4}{256\mu VL}\,(p_r^2 - p_i^2) \tag{16}$$

Equation (16) represents the differential equation of motion for the flow through the system. However, the left side of equation (16) contains the average pressure, $\bar{p}$, as defined by equation (13). The solution of equation (16) by numerical integration becomes rather involved if the term $\frac{d\bar{p}}{dt}$ is used. In order to simplify equation (16) the term $\frac{d\bar{p}}{dt}$ is replaced by $\frac{dp_r}{dt}$ and is justified in Appendix A for pneumatic system designs which are most prevalent at present. Equation (16) is rewritten as

$$\frac{dp_r}{dt} = \frac{-\pi\, d^4}{256\mu VL}\,(p_r^2 - p_i^2) \tag{17}$$

Equation (17) is now altered by the substitution of

$$K = \frac{\pi\, d^4}{256\mu VL} \tag{18}$$

and is finally given as

$$\frac{dp_r}{dt} = K(p_i^2 - p_r^2) \tag{19}$$

EXPERIMENTAL APPARATUS AND PROCEDURE

The experimental phase of this program was carried out in two parts. The first tests simulated the ascent trajectory of multi-stage missiles wherein the ignition of successive stages was simulated by impulse-type inputs. The final series of tests were run using continuous-type inputs for the ascent and descent of single stage missiles as well as for bombs dropped from reasonably high altitudes.

Impulsive-Type Inputs

A schematic drawing of the experimental apparatus for the impulse pressure inputs is shown in Figure 2. The system consisted of a vacuum tank(36,000 cubic inches) connected in series with a diaphragm section, the test tubing and a sensing volume. The sensing volume (1.7 cubic inches) for these tests was actually the volume of the differential transducer used to measure the pressure difference between the input and response ends of the test tubing. The tubing tested had line lengths of 30, 45, 60, and 75 inches. Three diameters (1/16, 1/8, and 5/32 inch) were tested for each line length. As shown in Figure 3, four input pressure functions, i.e., four ascent trajectories, were run for each geometric configuration. The trajectories were simulated by means of four orifice plates, each containing four small holes. The hole sizes ranged from a minimum diameter of approximately 0.002 inch, up to a maximum diameter in the neighborhood of 0.020 inch. In order to simulate a given trajectory a particular orifice plate was chosen and the diaphragm section assembled in the following manner: On the vacuum tank, or downstream side, of the orifice plate, two sheets of polyester film were located. Between these films four separate helical-coiled platinum wires were located immediately in front of each hole in the orifice plate. The ends of the platinum coils were soldered to copper leads, through which current could be passed. In order to

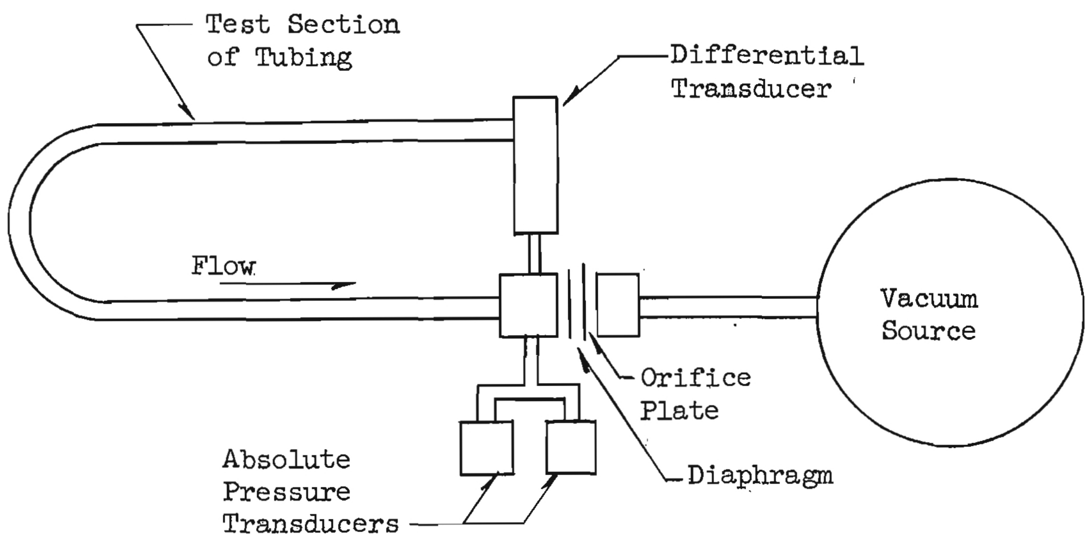


Fig. 2 Schematic of Apparatus Used To Produce Impulsive Input Pressure Functions For Ascent Trajectories.

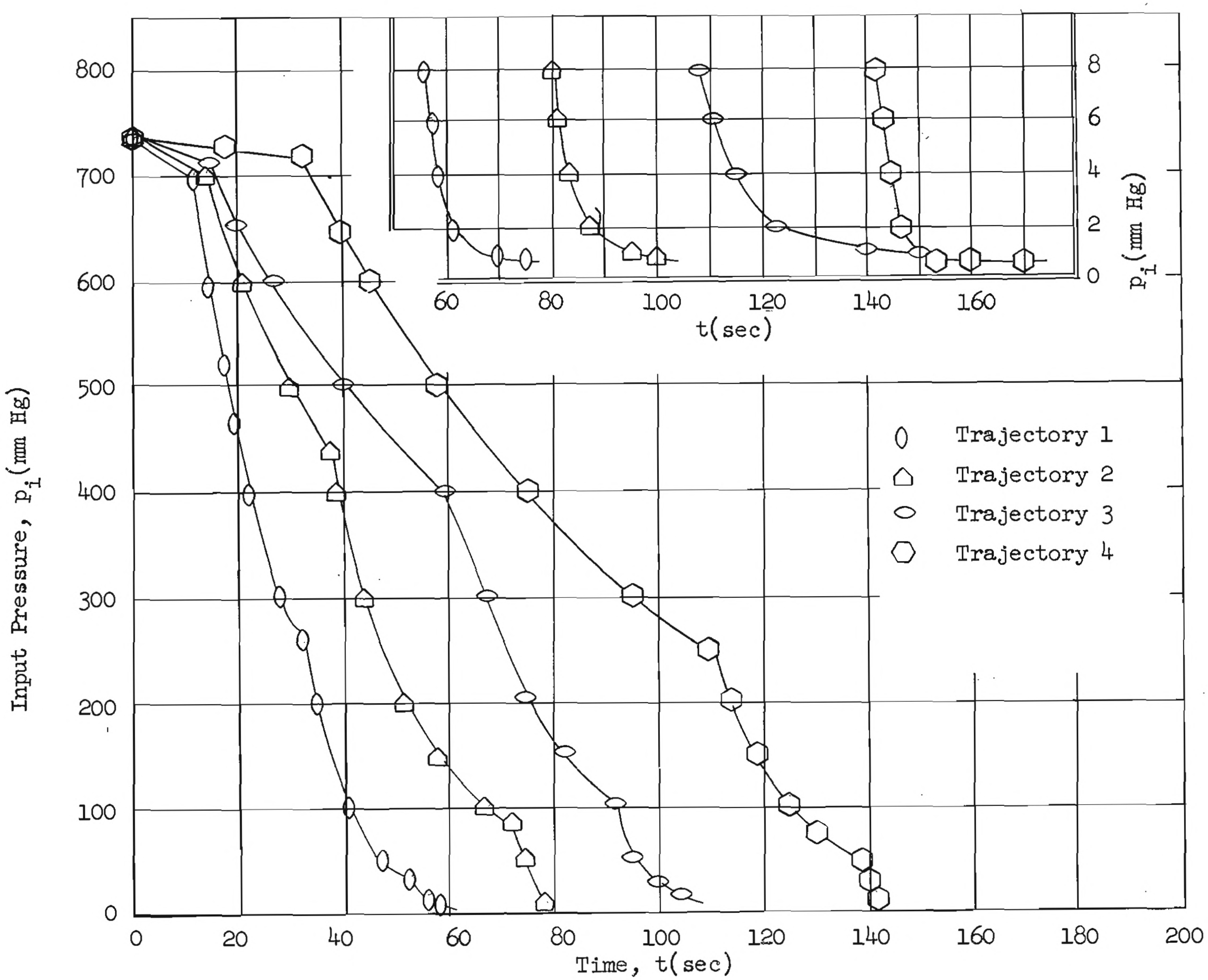


Fig. 3 Typical Impulsive Ascent Trajectories

simulate a given trajectory, the platinum coils were individually energized with 12 volts AC at predetermined times. As each platinum coil was energized, the film covering an orifice hole was burned, thus allowing the air in the test system to flow in the direction of the vacuum tank. Cross-flow between orifice holes on the downstream side of the orifice plate was prevented by encircling each platinum coil with an O-ring, which was seated in the face of the downstream flange. A simple system of relays was devised for firing the first two platinum coils. The first platinum coil was fired automatically when the relay circuit was energized. The firing time of the second platinum coil could be selected by adjustment of selector switches for the second relay in the circuit. Coils three and four were fired manually at selected times, which were observed on an electric stop clock. The combined relay system and electric stop clock gave repeatability to within one-tenth of a second for any given set of selected firing times. The polyester film actually served two purposes. First, it essentially served as a valve for each hole in the orifice plate and second, the film acted as insulator for the copper leads to which the voltage was applied. In order to simulate the trajectories shown in Figure 3, it was necessary to fire the platinum coils in a manner such that the smallest hole in the orifice plate fired first and then the remaining holes fired in order of ascending size. As each hole fired, an impulse resulted, since the effect of opening a hole was similar to that of a small shock-tube. The resulting discontinuities in the derivative $(\frac{dp}{dt})$, as shown in Figure 3, are assumed to simulate the firing of successive stages in a multi-stage missile system. The equilibrium pressures for the ascent trajectories varied between one and one and one-half millimeters of mercury absolute depending on the mass of atmospheric air in the test line and sensing volume at the start of a run. That is, the smaller diameters and shorter line lengths resulted in the lowest absolute equilibrium pressures.

Continuous-Type Inputs

The continuous-type input pressure functions were run for ascent and descent trajectories. A schematic of the experimental apparatus is shown in Figure 4. The system is essentially similar to that used for the impulse-inputs, with the exception that the diaphragm section was replaced by a cam-piston arrangement. The operation of the cam-piston arrangement consisted of experimentally determining the cam shape such that the flow through the orifice in the piston would simulate the desired trajectories. The movement of the cam was obtained by attaching the cam to a rack, which was in turn geared to a speed reducer driven by a synchronous motor. The use of a synchronous motor ensured a constant speed drive system. As can be seen in Figure 5, the air was allowed to flow from (descent) or to (ascent) the tankage by means of a tube located in the left side of the piston case. The piston is seen to be divided into three sections by means of O-rings. Two O-rings are located to the left to seal the cylinder from the atmosphere. A second O-ring (to the right) prevents the flow of air from the tankage to the test system. Thus, the only manner in which air can flow between the tankage and test system, is through the inside of the piston and up through the piston orifice. The flow actually starts when the piston orifice is moved to the right beyond the O-ring seal. Five descent trajectories (Figure 6) and three ascent trajectories (Figure 7) were run for line lengths of 30, 45, 60, and 75 inches, inside tube diameters of 1/16, 1/8, 5/32, and 0.186 inch, sensing volumes of 1.7, 5, 10, 20, and 40 cubic inches, and all combinations thereof.

Pressure Measurements

The input pressures were measured by means of six absolute transducers. The pressure difference between the input end of the tube and the sensing volume was measured by means of a differential transducer. The outputs from the transducers were amplified and then recorded by means of a multi-channel oscillograph, using seven inch photographic paper. Table 1 gives the range over which each of the absolute transducers was used. The use of overlapping transducer ranges was necessary

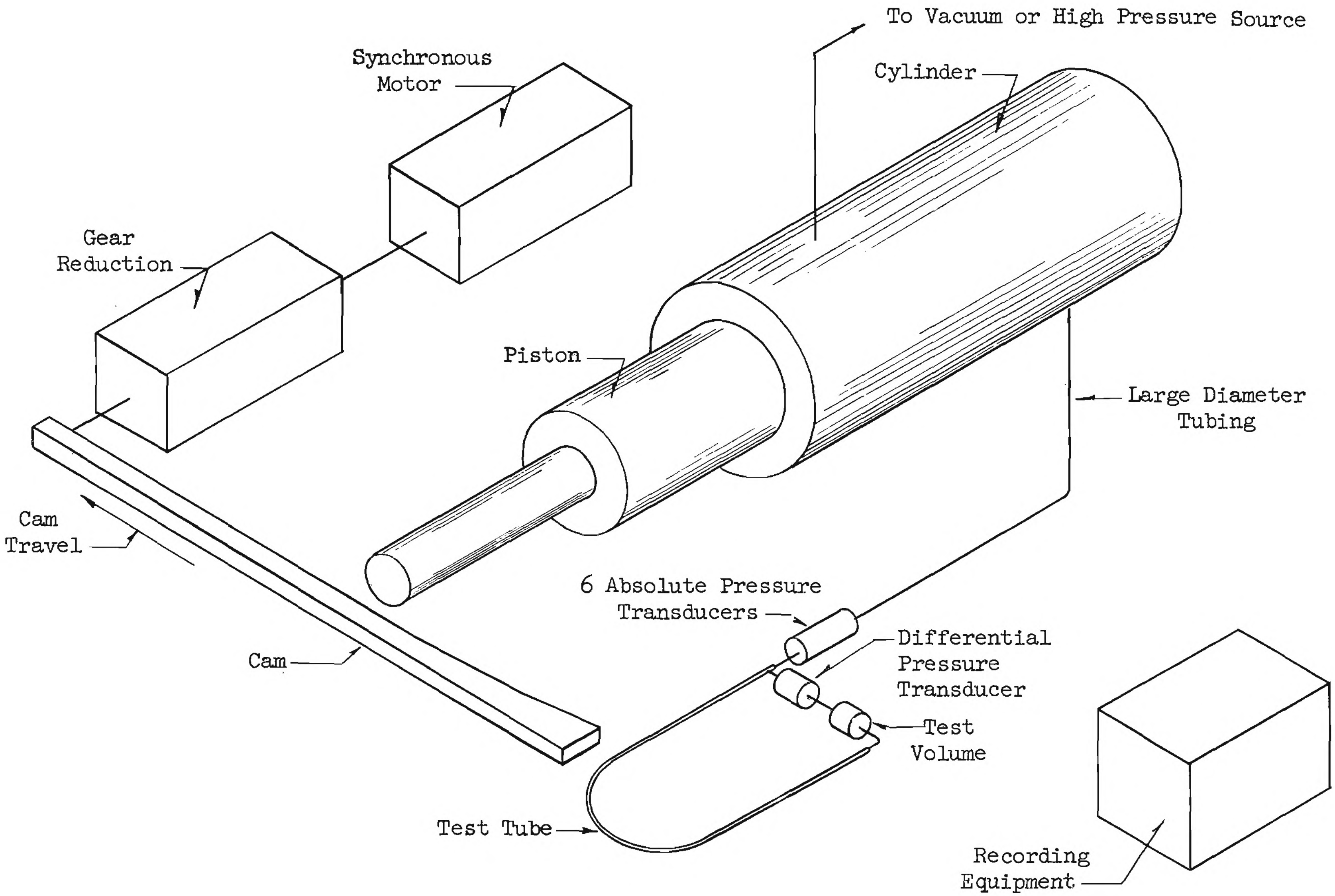


Fig. 4 Schematic of Apparatus Used to Produce Continuous Input Pressure Functions

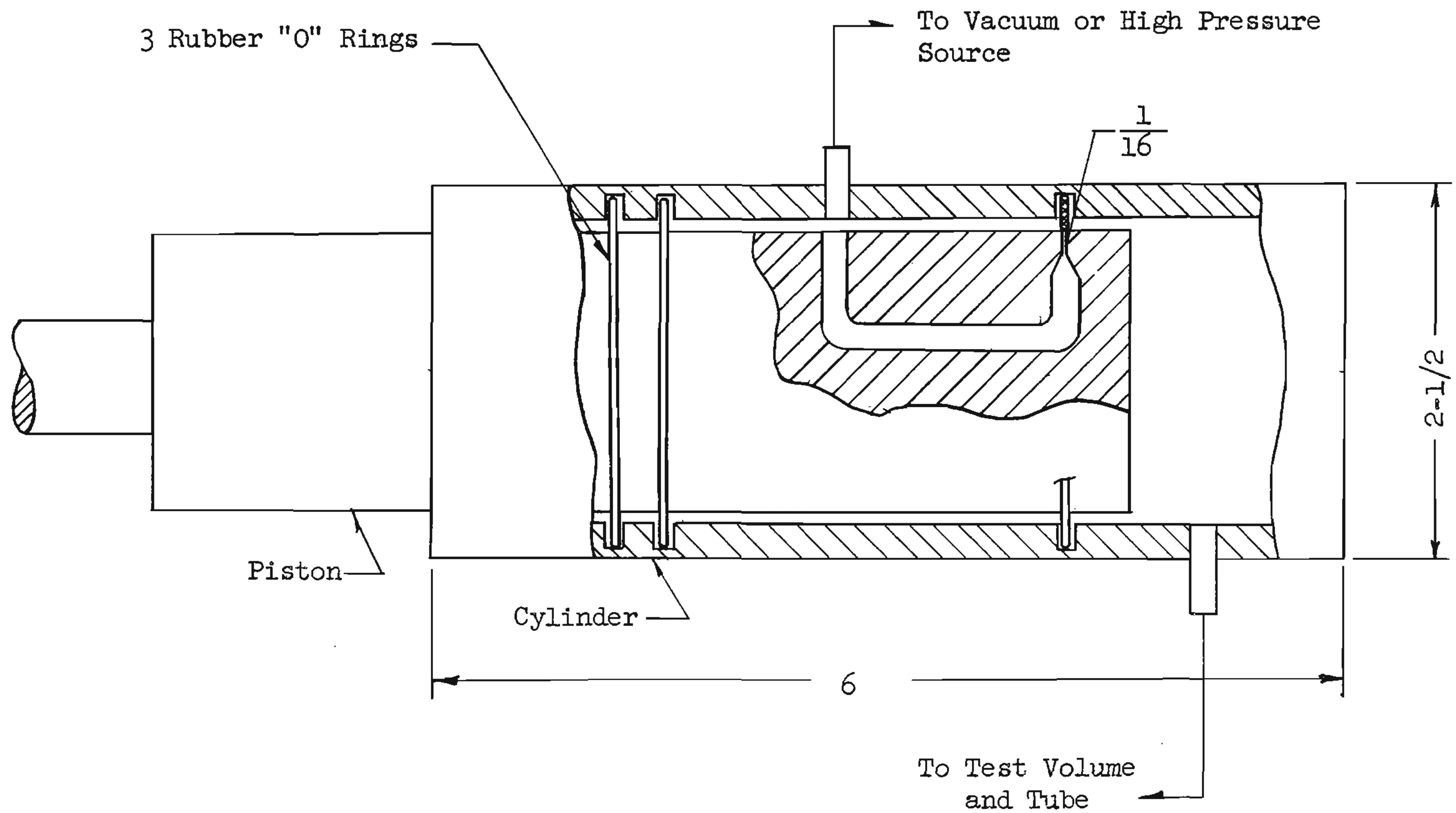


Fig. 5 Detail of Piston and Cylinder Assembly

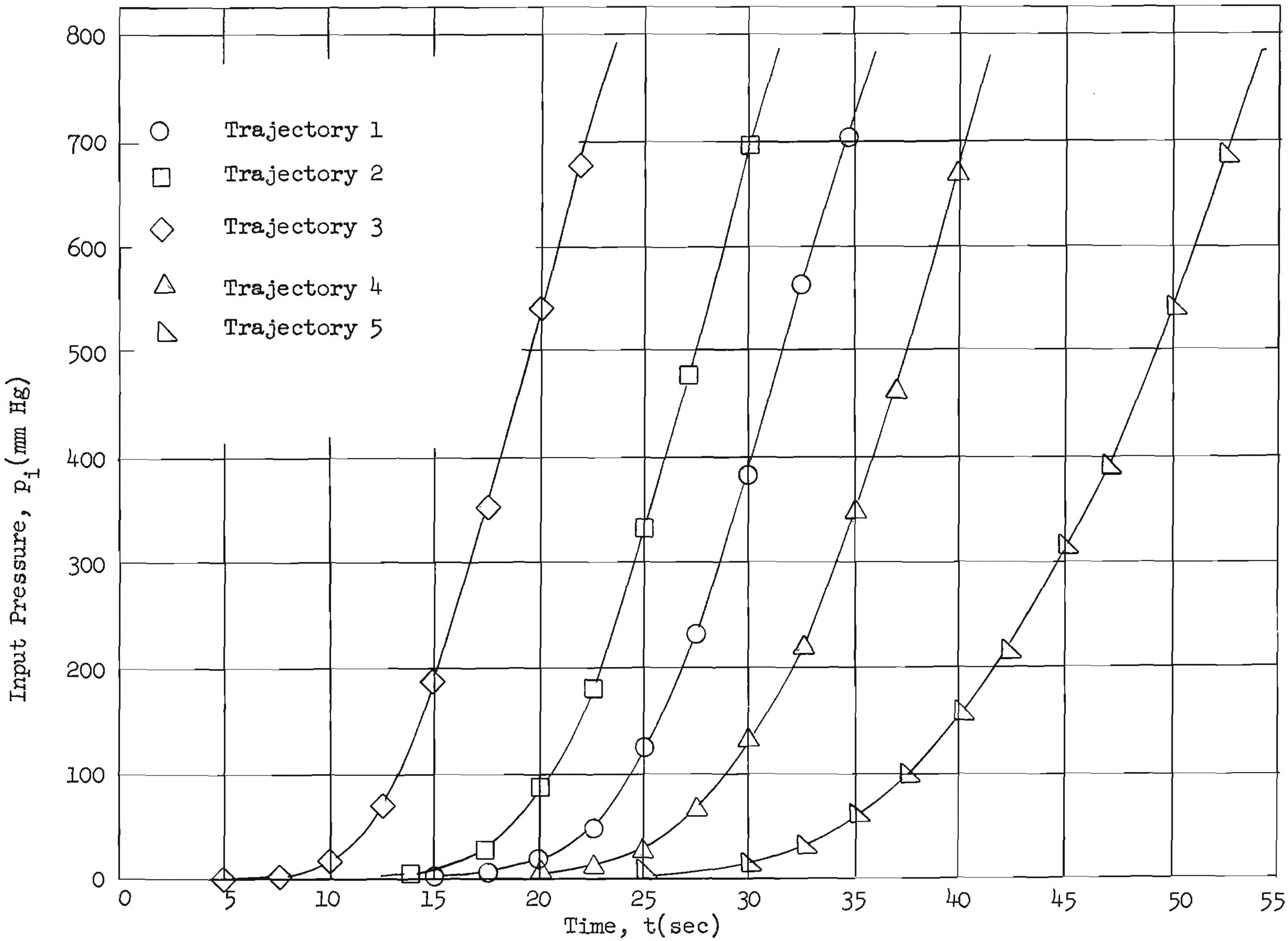


Fig. 6 Typical Continuous Descent Trajectories

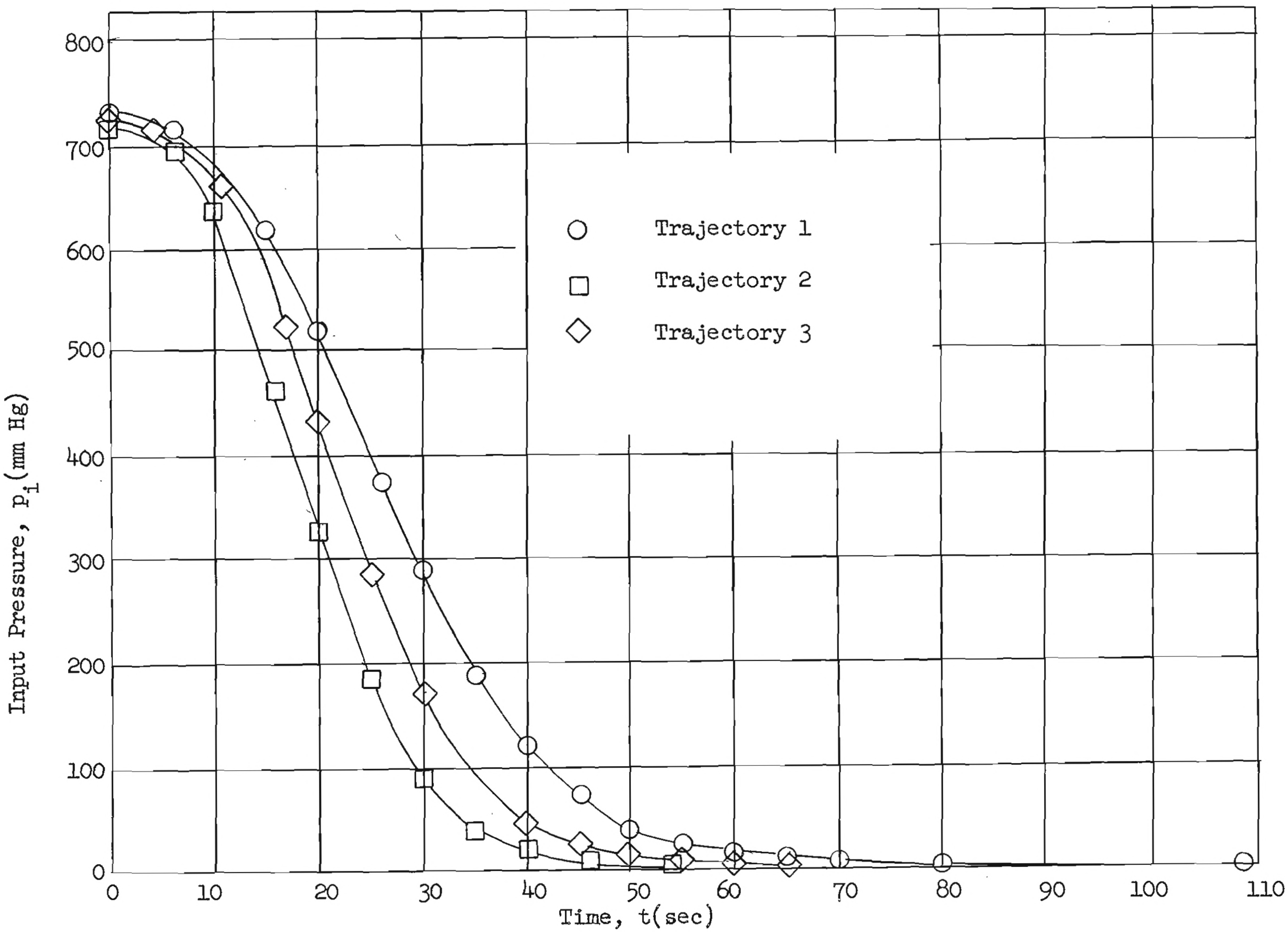


Fig. 7 Typical Continuous Ascent Trajectories

TABLE I

Pressure Ranges Measured by
Absolute and Differential Pressure Transducers

Absolute Pressure Transducers psia	Absolute Pressure Range Measured mm Hg.
0 - 15 A*	Atmospheric (740 mm approx.) - 300
0 - 15 B*	360 - 170
0 - 10	230 - 90
0 - 5 A*	100 - 30
0 - 5 B*	40 - 10
0 - 2	15 - 0

Differential Transducer psi	Differential Pressure Range mm Hg.
± 0.5	± 25

* Two 0 - 15 and two 0 - 5 absolute pressure transducers were used. The various ranges are indicated as A and B for clarity.

for added sensitivity and maximum accuracy with the available equipment. Figure 8 represents a typical oscillograph descent record, showing the overlap of the transducer traces and the useful range for each individual trace. The timing lines have been omitted in Figure 8 for reasons of clarity. The reduction of the data starts at Point I in Figure 8. This point represents the equilibrium vacuum pressure in the test system before the run commences. The equilibrium pressures between 0.25 mm Hg abs and 1.0 mm Hg abs. were monitored by means of an electronic vacuum gauge, which was calibrated against a McLeod gauge. The accuracy of this measurement is within 30 microns. Knowing Point I, the data is measured along the 0 - 2 transducer trace until Point II which now becomes the starting point, Point III, for the measurement of pressures along the 0 - 5 transducer trace. The 0 - 5 trace is then followed to Point IV, which, as before, becomes the value of the starting Point V for the 0 - 10A trace and so on for the remaining traces. Using this method of data production and reduction resulted in errors estimated to be less than one per cent of the absolute pressure at any level of absolute pressure. However, the gain in overall sensitivity and accuracy more than compensated for this one per cent error. Ascent data was reduced in an identical manner in that the data was reduced starting at the lowest equilibrium vacuum pressure and working backward on the oscillograph charts.

The differential pressure between the input and sensing volume was measured by means of a $\pm$ 0.5 psi differential transducer. A typical differential trace is shown plotted in Figure 8.

Calibration

The calibration of the absolute pressure transducers was accomplished with the use of four absolute pressure gauges having ranges of 0 - 20, 0 - 100, 0 - 400, and 400 - 800 mm Hg absolute. The absolute pressure gauges had previously been calibrated against an Arrowsmith gauge. The transducers were calibrated, using pressure variations in the same direction as would be encountered during the experimentation, thus the hysteresis errors were minimized. Calibrations were run each day prior to the actual experimental runs.

The differential transducer was calibrated by use of an alcohol manometer, sensitive to ± 1 mm alcohol. During the initial calibrations of the differential transducer, it was determined that the calibration was essentially invariant with absolute pressure level. Thus, the later calibrations were made, using approximately room pressure as the reference pressure.

Accuracy of Measurements

The error in the experimental data is essentially that of accuracy and repeatability of calibrations from day to day plus the human error in reading the oscillograph records. The maximum errors in the absolute pressure gauges are estimated to be of the order of ± 1/2 per cent of full scale. The calibrations of the absolute pressure transducers indicate that the maximum deviations from the mean are also of the order of ± one half of one per cent of full scale. On the basis of these errors, it is estimated that the maximum error for the absolute transducers was of the order of ± one per cent of the range used for each transducer. The differential calibration: ± one half of one per cent of full scale, alcohol manometer: ± 0.06 mm Hg, reading error ± one-quarter of one per cent of full scale. Table II is presented as a summary of the estimated errors in the experimental data.

TABLE II

Estimated Pressure Errors

Transducer	Pressure Range mm. Hg.	Estimated Pressure Error mm. Hg.
0 - 15 A	740 - 300	± 7
0 - 15 B	360 - 170	± 7
0 - 10	230 - 90	± 5
0 - 5 A	100 - 30	± 2.5
0 - 5 B	40 - 10	± 2.5
0 - 2	15 - 0	± 1
Differential	± 25	±.2

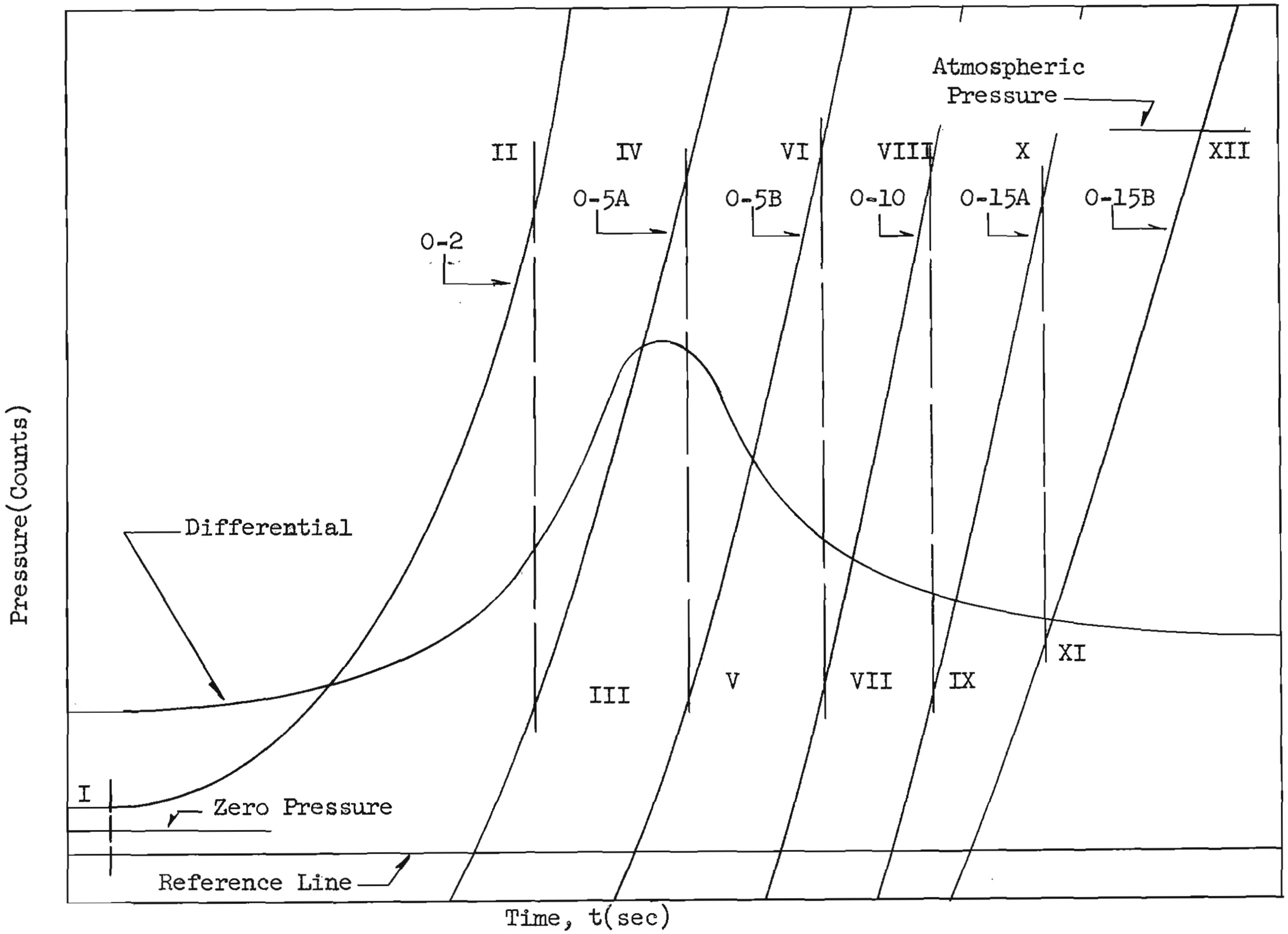


Fig. 8 Typical Descent Oscillograph Record

RESULTS

The results of this analysis apply to geometrical configurations discussed previously and for continuous and impulse type input functions as described by Figures 3, 6, and 7. In order to attempt to correlate the theory with experiment, the parameter K, (Equation 19), was empirically determined from the experimental data. The procedure consisted of taking the measured data from a given run (i.e. a given geometric set-up and a given trajectory) and computing K, using Equation (19) at randomly selected time points. For any given run, at least five values of K, corresponding to five different time points, were computed. An arithmetic average of the empirically determined K values was then computed. The volume used in computing K was taken as the total volume of the system (i.e. the arithmetic sum of the line volume and total sensing volume, including the volume of the differential transducer). Figure 9 represents a plot of K versus $\phi = \dfrac{L\ V}{10^4\ d^4}$. Each circled point in Figure 9 represents the arithmetic mean of at least five empirically determined values of K for a given geometric set-up and a given trajectory. The computation of K was found to be a function ϕ alone, i.e., independent of the type of trajectory (ascent or descent), the trajectory rate $\dfrac{dp_1}{dt}$, and the type of input (continuous or impulse) for the range of variables tested. The solid line in Figure 9 represents a faired curve through the empirically determined values of K, whereas the broken curve represents values of K as computed from Equation (19) wherein

$$K = \frac{\pi d^4}{256 \mu VL} \qquad \text{or} \qquad K = \frac{9.4}{\phi}$$ (where K has the units of 1/mm-sec).

The equation of the curve faired through the experimental data has

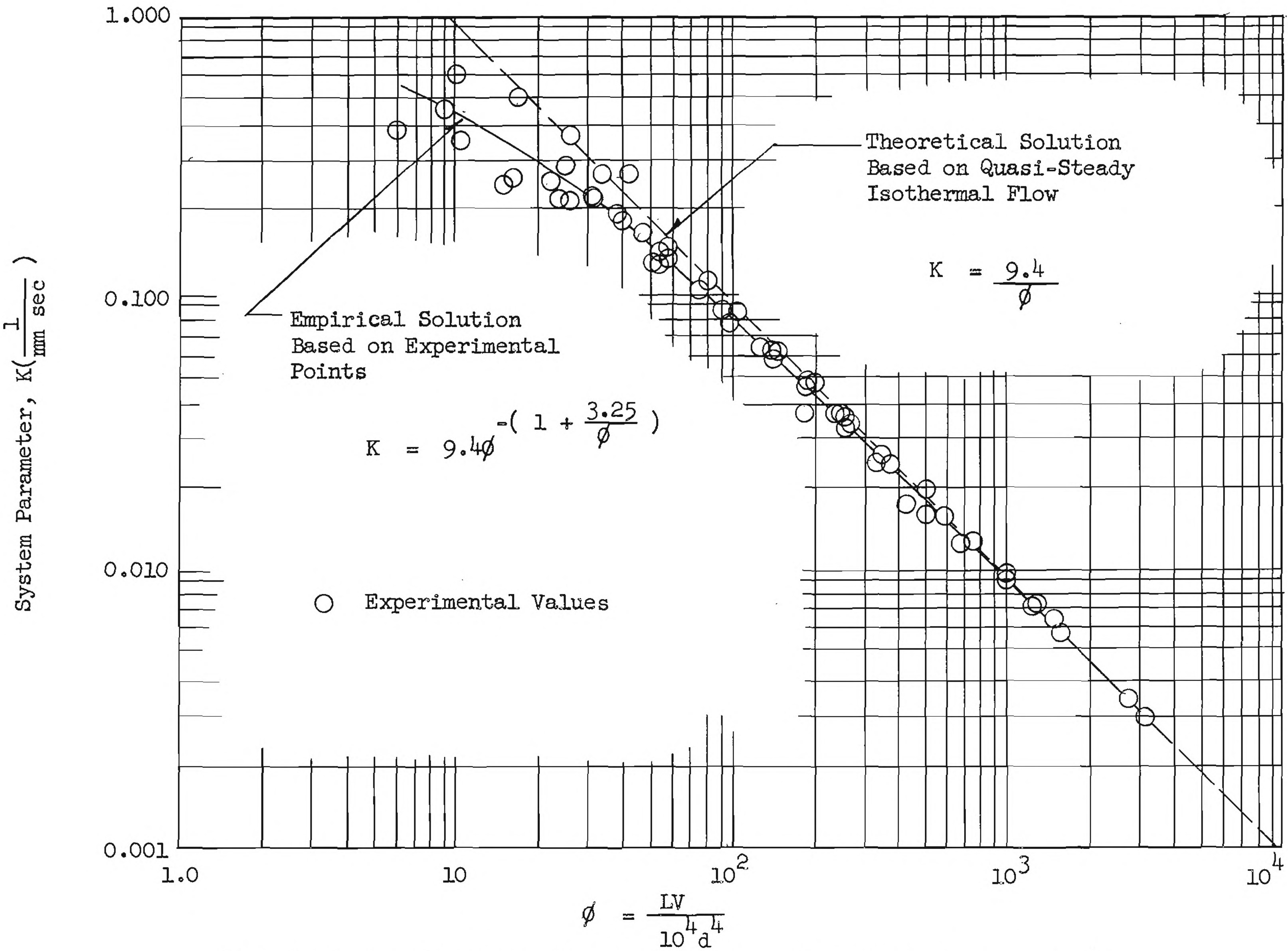


Fig. 9 System Parameter, K, as a Function of Geometric Parameter, ϕ

been determined to be $K = 9.4\phi^{-(1 + \frac{3.25}{\phi})}$. From Figure 9 it is noted that the theoretical and empirical values of K agree very closely for values of $\phi \gtrsim 400$, however, as the value of ϕ decreases below 400 the theoretical and empirical values of K start to deviate with the deviation becoming considerably larger as ϕ is reduced to 10 (its lowest limit for these experimental tests). Since ϕ is a strong function of line diameter, the small values of ϕ can be associated with the large line inside diameters and intuitively small pressure lags. For the large values of ϕ the pressure lag is increased considerably, since in this case a small line inside diameter corresponds to the increased value of ϕ . The deviation between the empirical and theoretical values of K as ϕ is reduced is thought to result from the breakdown of the fully-developed flow assumption. Since small values of ϕ can be associated with large line inside diameters, then, for two lines of the same length the configuration with the small L/D would tend to violate the idea of a fully-developed flow over the entire tubing length. The result would be a larger pressure drop, which is indicated by the empirical value of K being lower than its theoretical counterpart. This arrangement becomes obvious if Equation (19) is rewritten as

$$K = \frac{\frac{dp_r}{dt}}{(p_i + p_r)(p_i - p_r)} \tag{20}$$

Thus, for a given value of $\frac{dp_r}{dt}$ and a given level of pressure in the system, $(p_r + p_i)$, the value of K is inversely proportional to the pressure drop, $\Delta p = p_i - p_r$. If the flow is developed over a small portion of the tubing, the Δp is increased and the result is a smaller value of K which is consistent with the comparison of theory and experiment shown in Figure 9.

The correlation of theory with experiment is shown in Figures 10- 18 wherein plots of Δp (pressure lag in millimeters) versus time are plotted. In order to evaluate Equation (19) the following procedure was used:

<u>Case I</u> Given $p_i(t)$: To Find $p_r(t)$.

Equation (19) is written as before as:

$$p_r' = \frac{dp_r}{dt} = K(p_i^2 - p_r^2) \tag{19}$$

and the initial conditions are given as:

$$t = 0 \qquad \begin{aligned} p_i &= p_r \\ p_r' &= 0 \end{aligned} \tag{21}$$

The value of K in Equation (19) is taken from Figure 9. In order to solve for $p_r(t)$ a two-term Taylor expansion is used, i.e.,

$$p_r(t+h) = p_r(t) + hp_r'(t) \tag{22}$$

where h is the time increment. The step-by-step integration is then carried out for the determination of $p_r(t)$. Having $p_i(t)$ and $p_r(t)$ leads immediately to $\Delta p(t) = p_i(t) - p_r(t)$.

<u>Case II</u> Given $p_r(t)$: To Find $p_i(t)$

In order to solve for $p_i(t)$ Equation (19) is rewritten as:

$$p_r' = K\,\Delta p(2p_r + \Delta p) \qquad (23)$$

Knowing $p_r(t)$, a plot of $p_r(t)$ versus t is constructed and values of $p_r'(t)$ are graphically determined. K is chosen from Figure 9 for a particular geometry and Δp is obtained from the resulting quadratic equation in Δp. The solution of $p_i(t)$ is easily obtained from the relation $\Delta p = p_i - p_r$.

Figures 10 - 13 represent the correlation of theory with experiment for the impulse-type pressure inputs for the ascent trajectories. These graphs represent typical results for conditions wherein the pressure lag has its maximum and minimum values as well as median values for the range of variables examined. The correlation is seen to be extremely good with the deviations of experiment from theory being no greater than the experimental accuracy. Figures 14 - 18 represent typical correlations for the ascent and descent phases for the continuous-type input pressure functions. As in the case of the impulse-inputs, the agreement between theory and experiment is definitely acceptable.

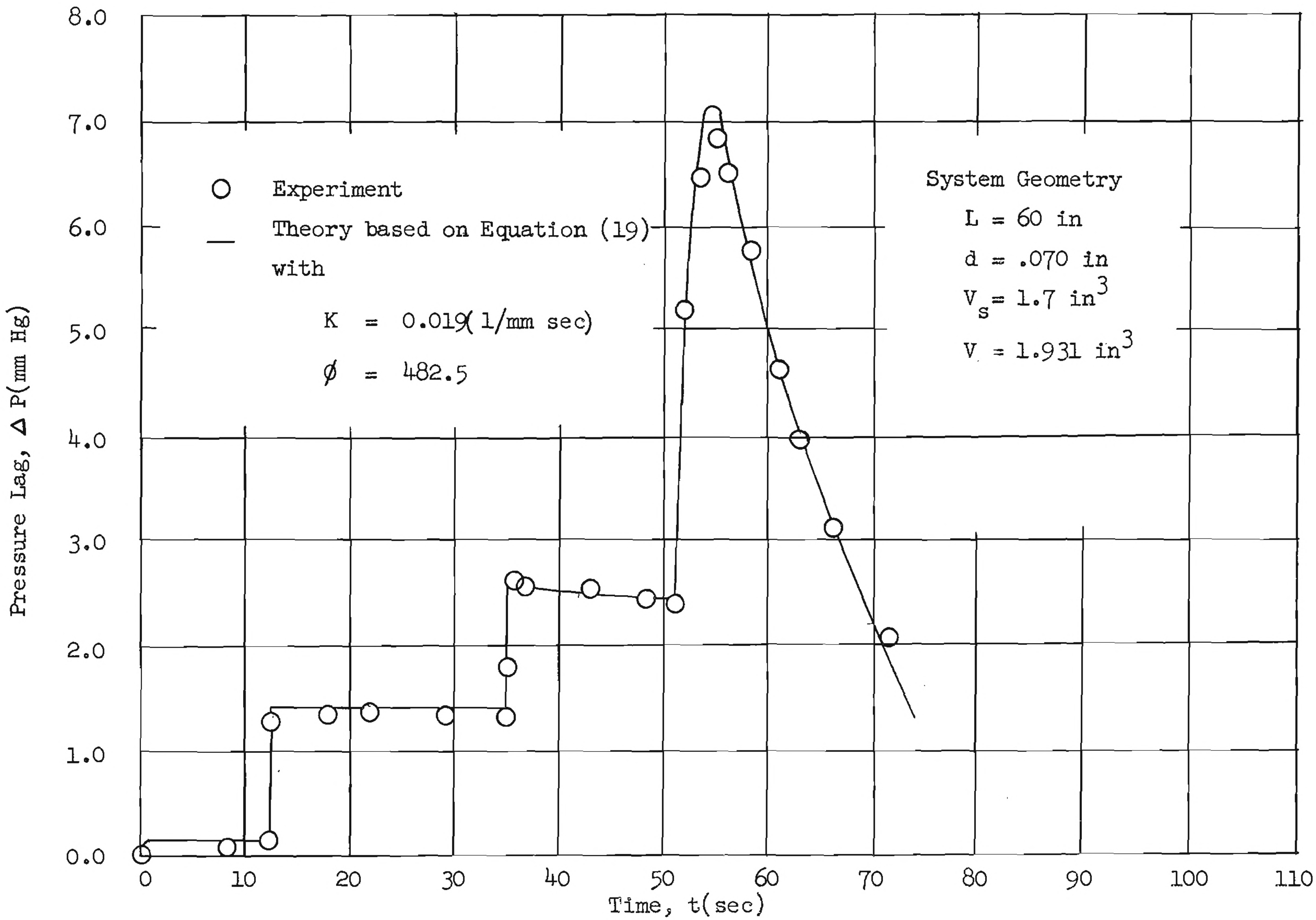


Fig. 10 Correlation of Experiment with Theory for Impulsive Ascent Trajectory No. 1

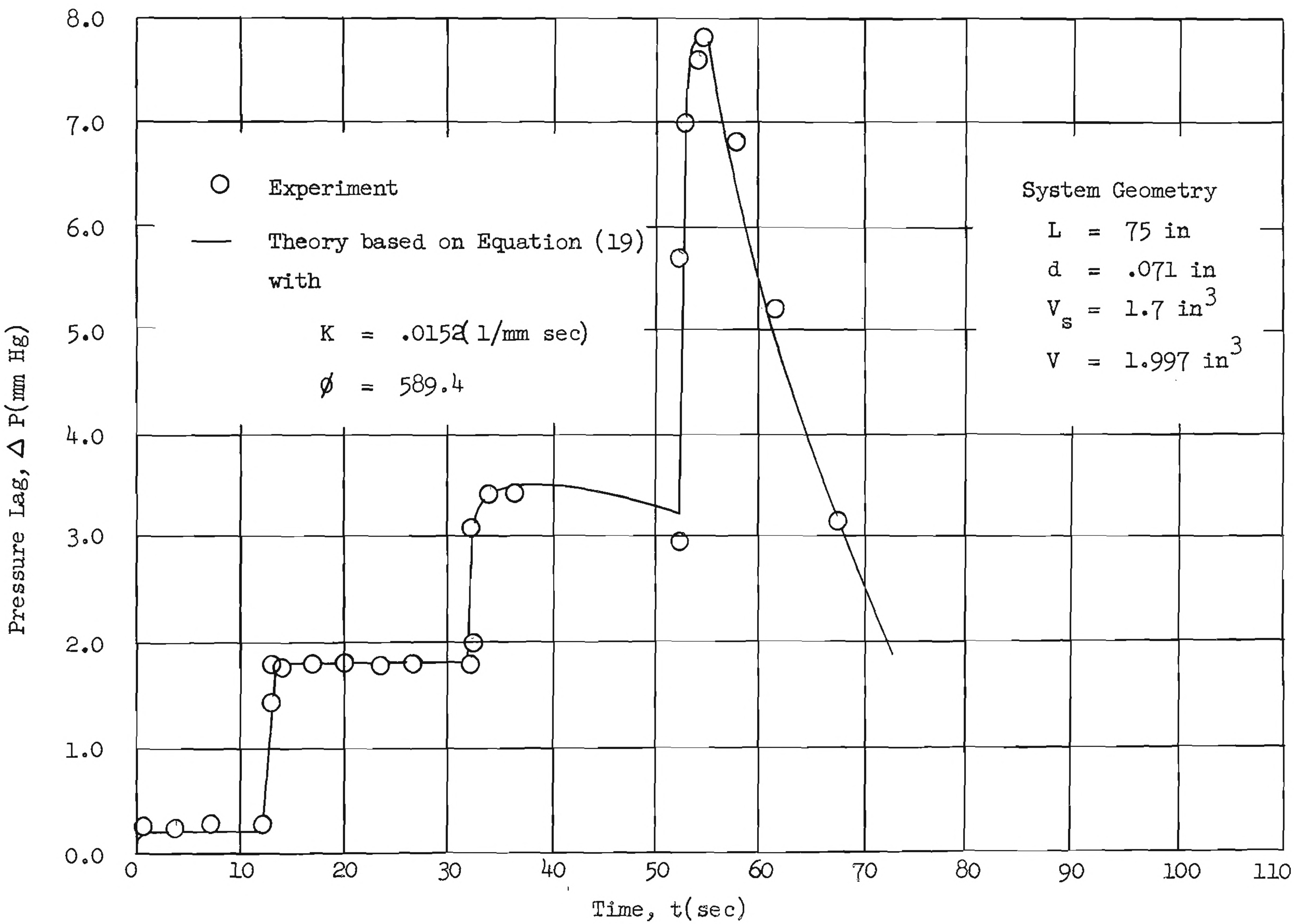


Fig. 11 Correlation of Experiment with Theory for Impulsive Ascent Trajectory No. 1

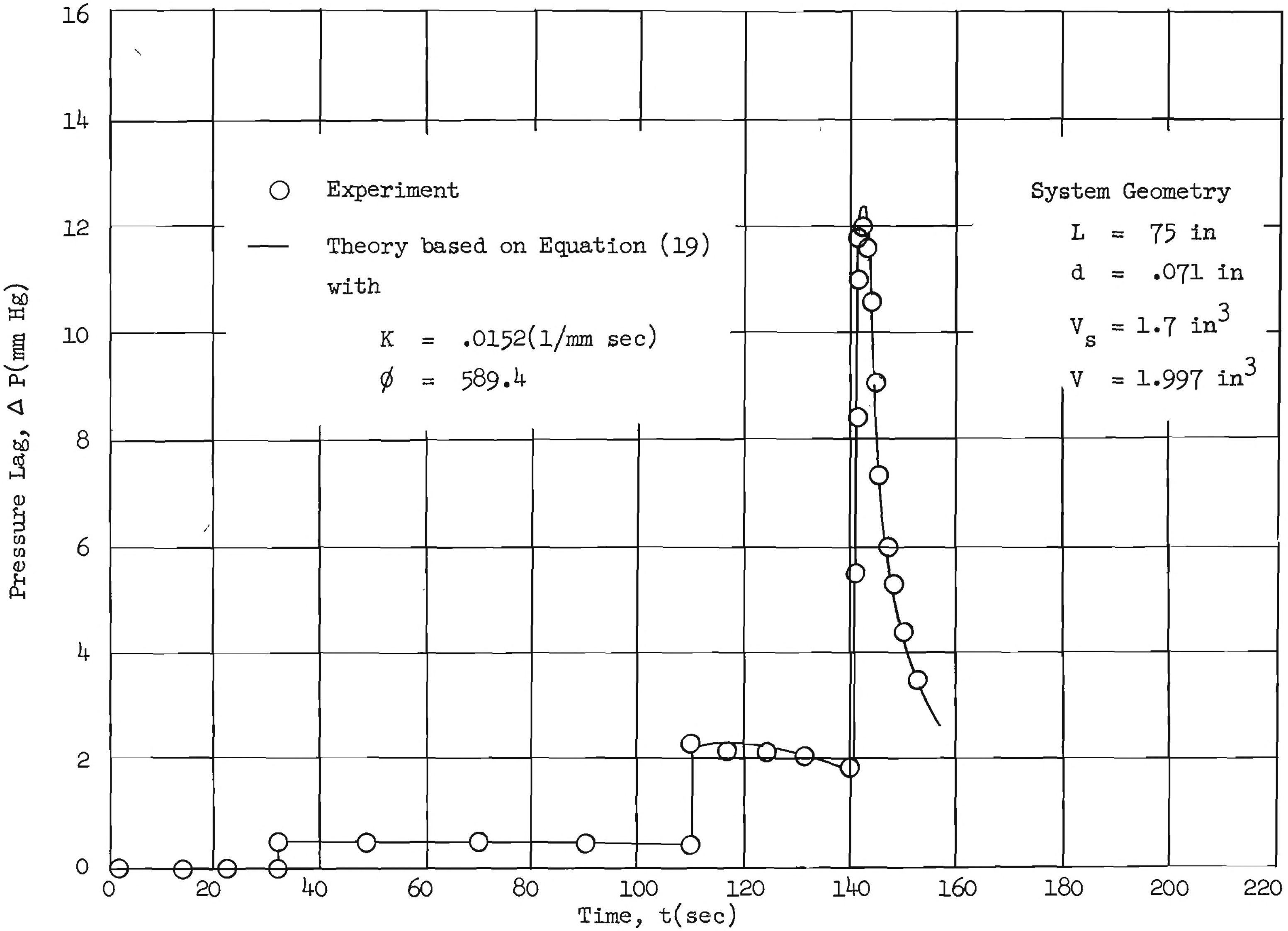


Fig. 12 Correlation of Experiment with Theory for Impulsive Ascent Trajectory No. 4

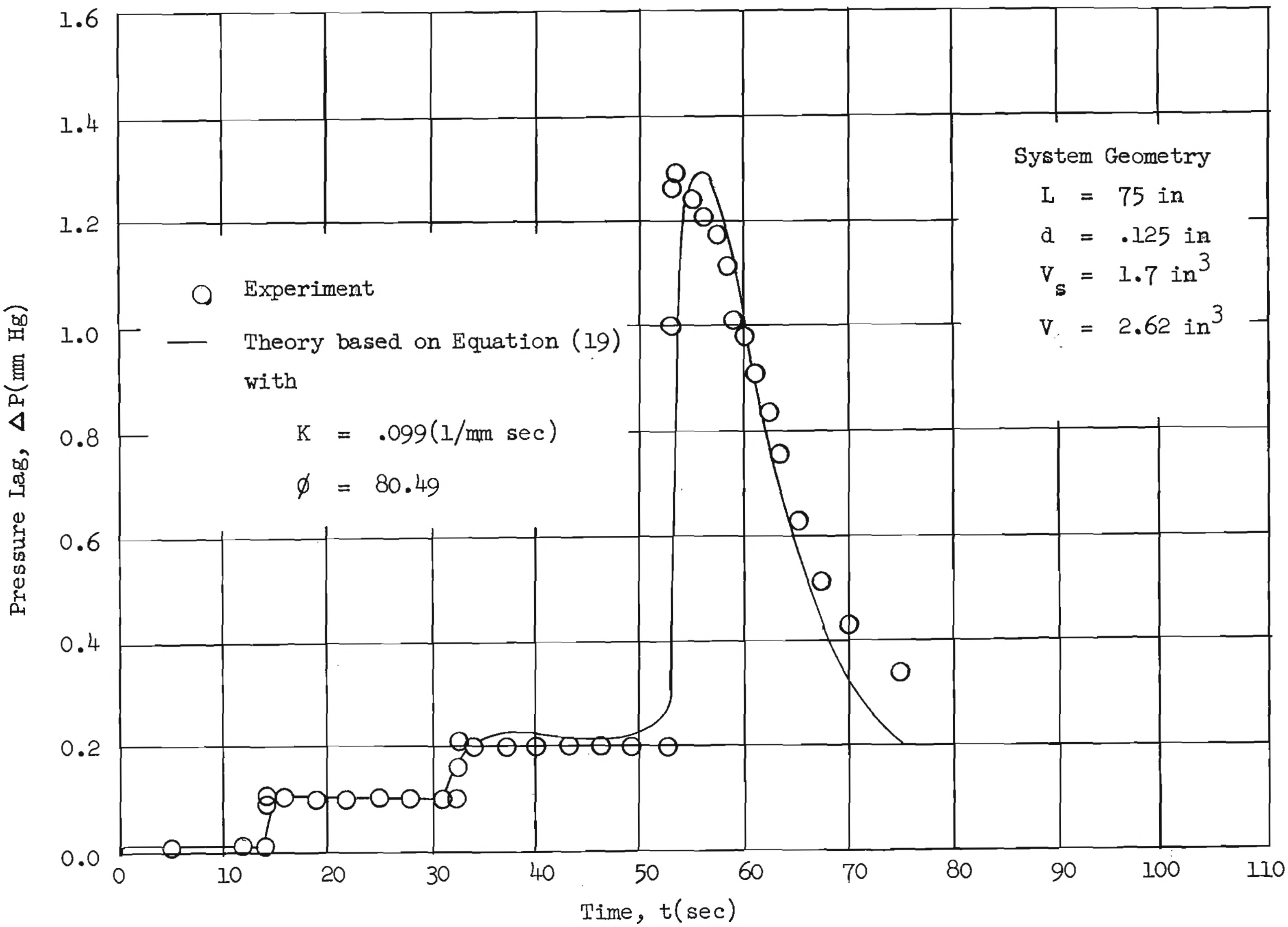


Fig. 13 Correlation of Experiment with Theory for Impulsive Ascent Trajectory No. 3

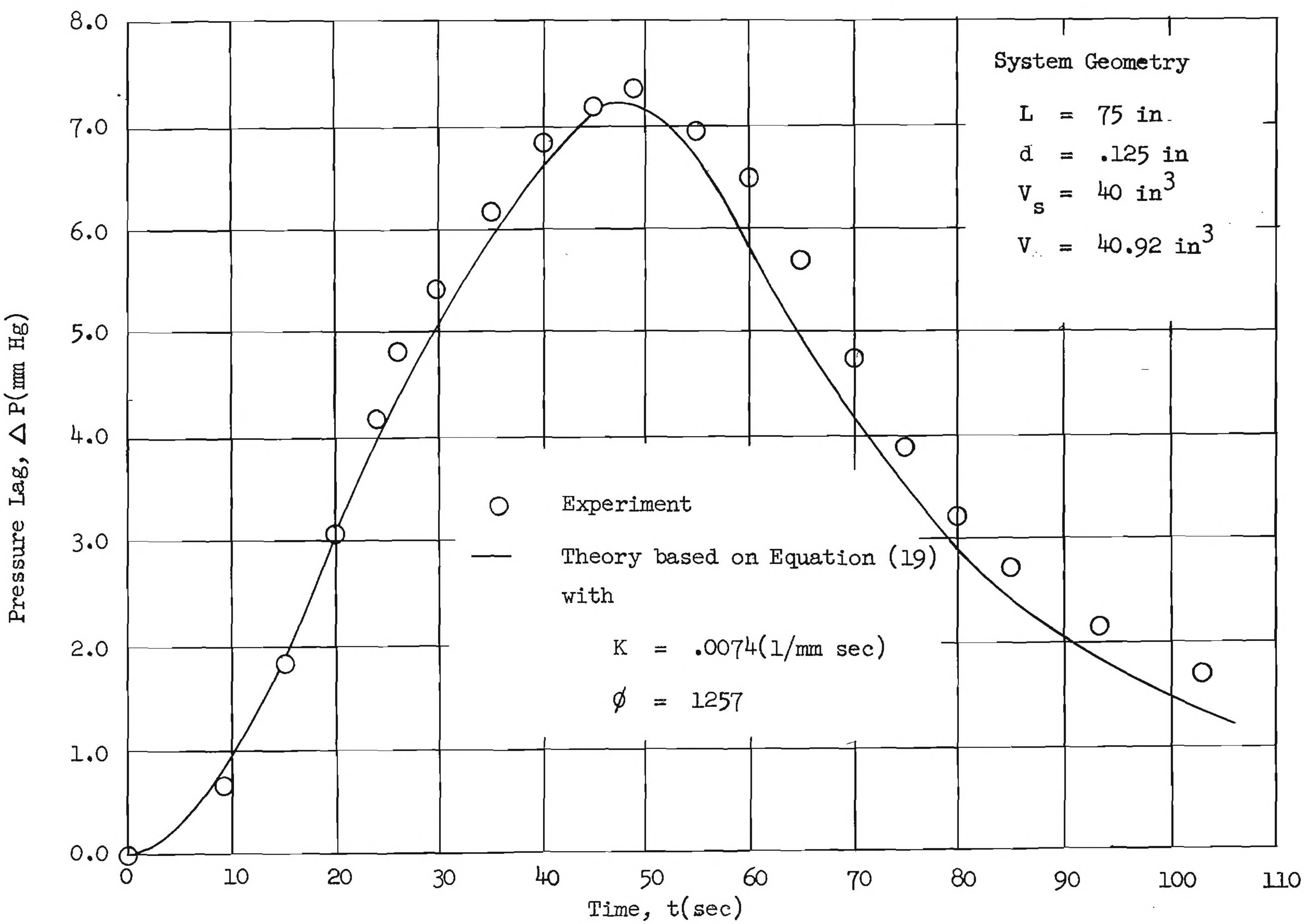


Fig. 14 Correlation of Experiment with Theory for Continuous Ascent Trajectory No. 3

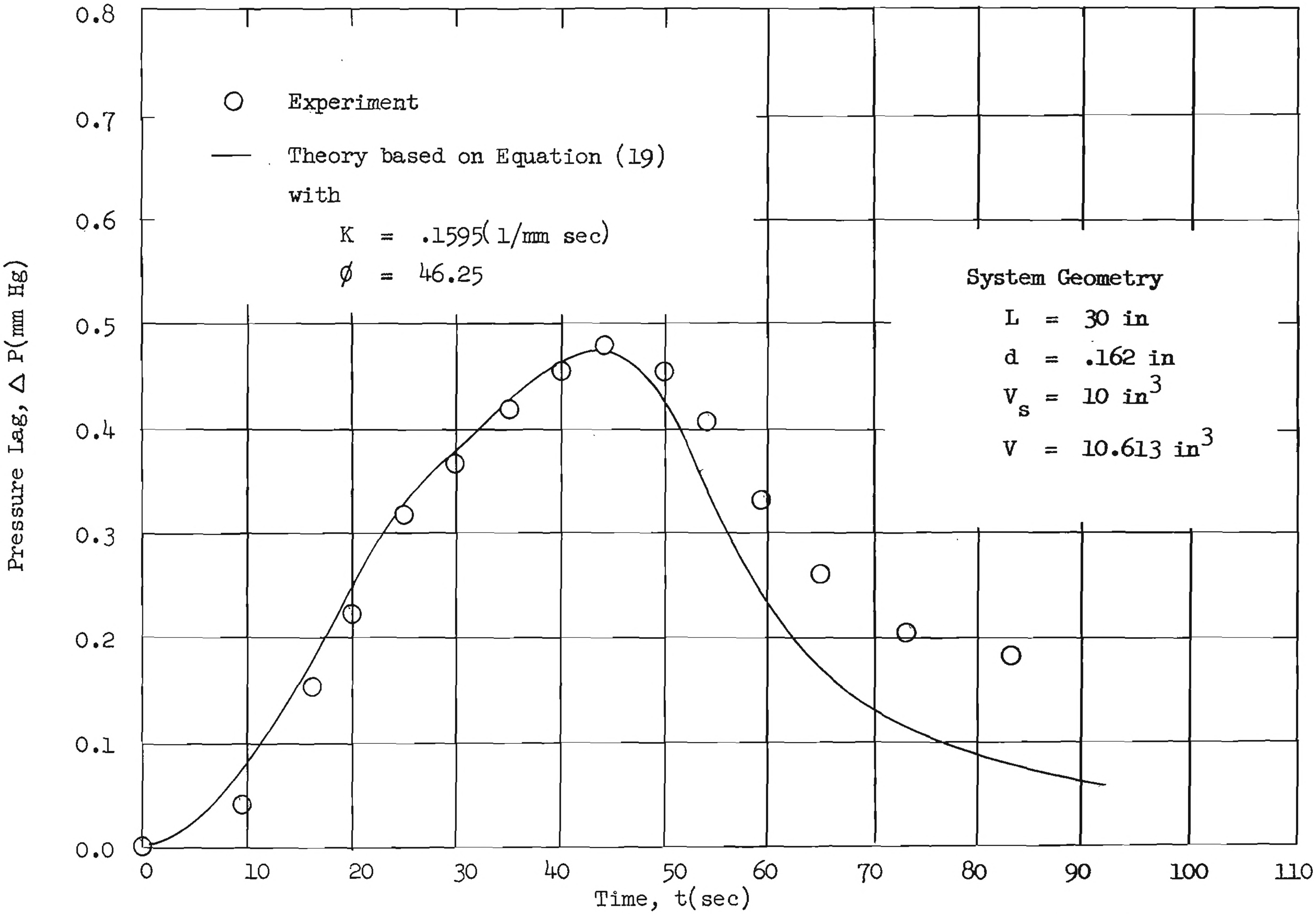


Fig. 15 Correlation of Experiment with Theory for Continuous Ascent Trajectory No. 2

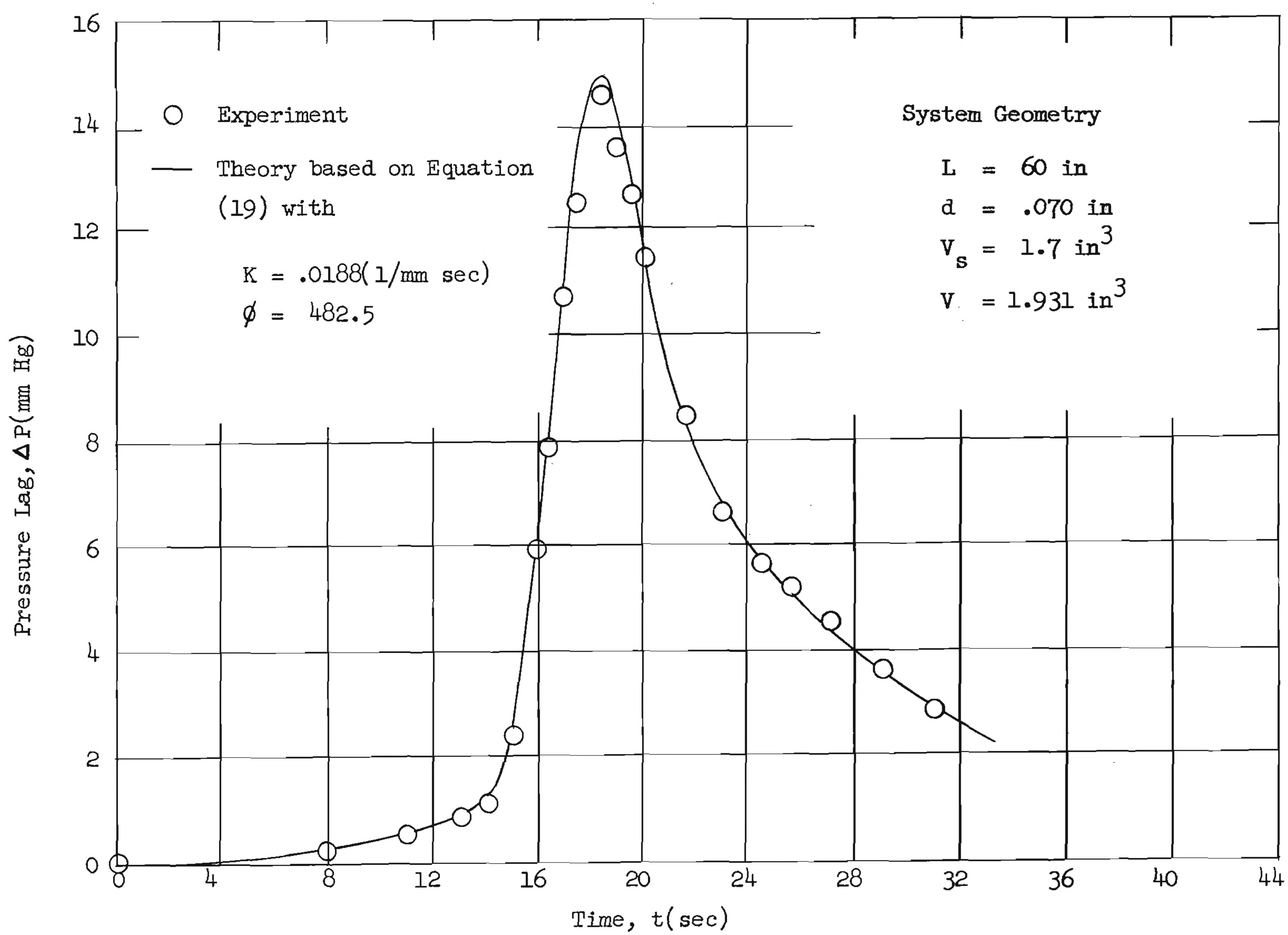


Fig. 16 Correlation of Experiment with Theory for Continuous Descent Trajectory No. 3

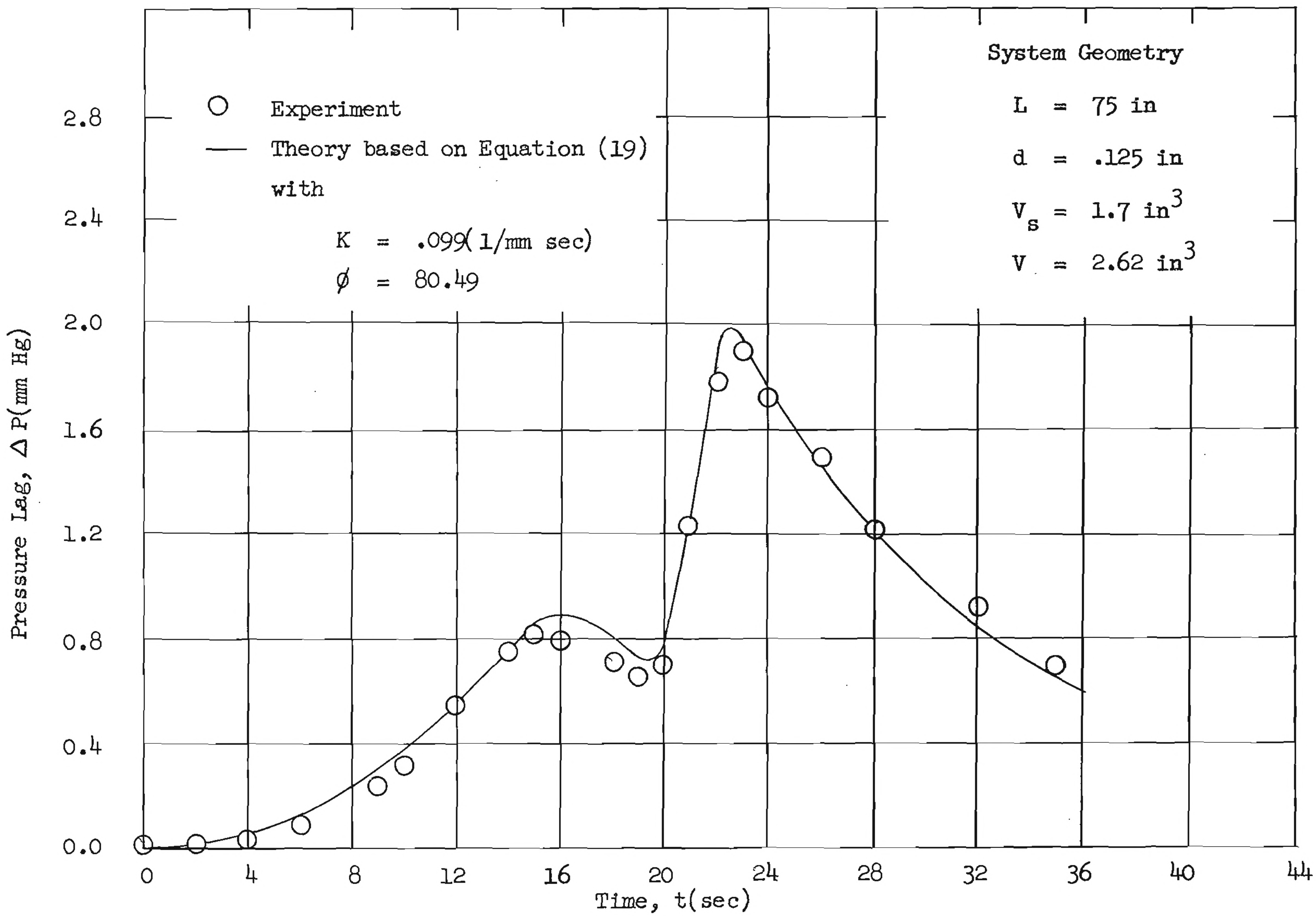


Fig. 17 Correlation of Experiment with Theory for Continuous Descent Trajectory No. 3

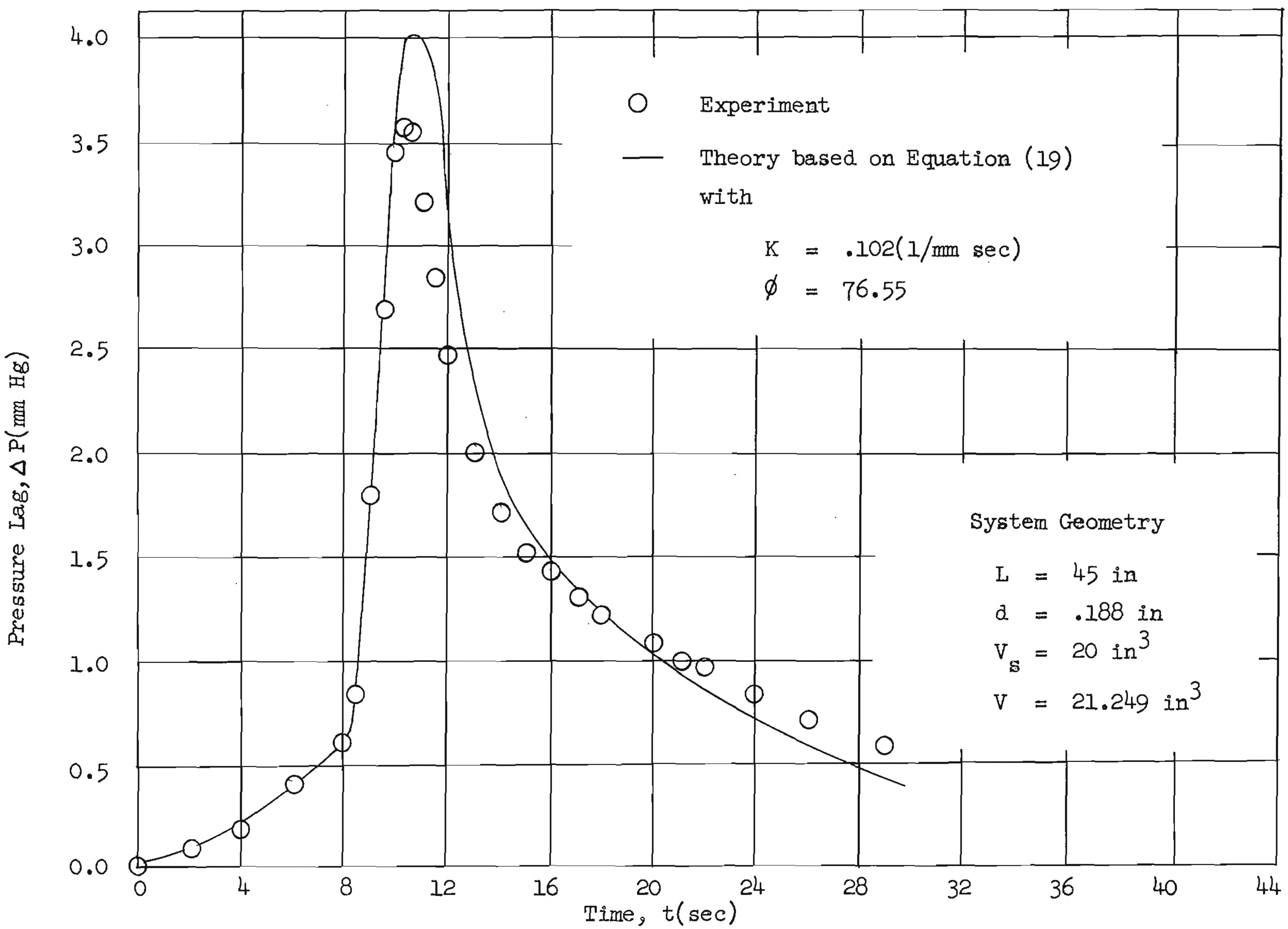


Fig. 18 Correlation of Experiment with Theory for Continuous Descent Trajectory No. 3

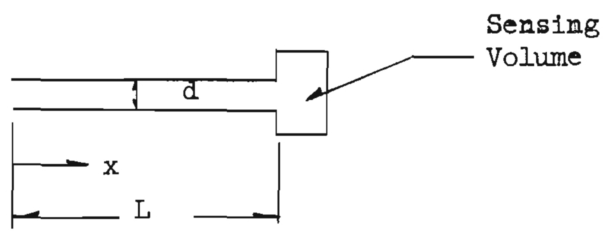
CONCLUSIONS

The results of this analysis are considered applicable for the prediction of pneumatic pressure lag in missile plumbing systems under the restriction of the range of parameters tested. The necessity of the empirical determination of the parameter, K , probably results from the failure of the fully-developed laminar flow assumption to extend the entire tube length for those configurations where L/d is small. The analysis shows that the parameter K is independent of trajectory and depends only on the system geometry. The definition of K indicates that the pneumatic lag for a given trajectory can be minimized if the geometric ratio, $(\frac{LV}{d^4})$, is minimized.

The effect of inlet temperature during the descent phase of a trajectory has not been examined. It is felt that the applicability of the present theory is questionable under the conditions of high inlet temperature during descending flight, since the assumption of isothermal changes of state would be violated over some inlet length of the tubing. In addition, the effect of connector fittings, whose inside diameter is smaller than the tube ID has not been determined. This latter effect can be instrumental in increasing the pneumatic lag, appreciably, since the fittings can act as a choke on the entire system. It is recommended that fittings be chosen such that the fitting ID is at least equal to the line ID if the pneumatic lag is to be minimized. A further consideration, which has not been investigated, concerns the effect of breathing on the boundary layer and shock or expansion pattern at the surface port. Observed experimental data indicates that outflow (ascent) and inflow (descent) can cause pressure measuring errors of the order of, or greater than, the pneumatic lag of the system itself.

APPENDIX A

In order to illustrate the justification of replacing $\frac{d\bar{p}}{dt}$ by $\frac{dp_r}{dt}$ in Equation (16), Equation (13) is evaluated.



Let V_s and V_t be the volumes of the sensing element and tubing ($\frac{\pi d^2}{4} L$), respectively, and also let the total volume, V, be the sum of $V_s + V_t$. Now according to Equation (10) the pressure at position x is given by

$$p^2 = (p_r^2 - p_i^2)\frac{x}{L} + p_i^2$$

Equation (13) is

$$\bar{p} = \frac{1}{V}\int_V p\,dV$$

and substitution for p is given by

$$\bar{p} = \frac{1}{V}\left[\int_{V_t} p\,dV_t + p_r V_s\right] \tag{A.1}$$

In Equation (A.1) it is assumed that the response pressure is uniform over the entire sensing volume.
Equation (A.1) is rewritten as

$$\bar{p} = \frac{1}{V}\left[V_t \int_0^1 p\, d\left(\frac{x}{L}\right) + p_r V_s \right] \qquad \text{(A.2)}$$

Substituting Equation (10) into Equation (A.2) gives

$$\bar{p} = \frac{1}{V}\left[V_t \int_0^1 \left\{ (p_r^2 - p_i^2)\frac{x}{L} + p_i^2 \right\}^{1/2} d\left(\frac{x}{L}\right) + p_r V_s \right] \qquad \text{(A.3)}$$

Integration of Equation (A.3) yields

$$\bar{p} = \frac{1}{V}\left[V_t\, (2/3)\, \frac{(p_r^3 - p_i^3)}{(p_r^2 - p_i^2)} + p_r V_s \right] \qquad \text{(A.4)}$$

Equation (A.4) is rewritten as

$$\bar{p} = \frac{1}{V}\left[(2/3)\, V_t \left\{ \frac{(p_r - p_i)(p_r^2 + p_r p_i + p_i^2)}{(p_r - p_i)(p_r + p_i)} \right\} + p_r V_s \right] \qquad \text{(A.5)}$$

or

$$\bar{p} = \frac{1}{V}\left[(2/3)\, V_t \left\{ (p_r + p_i) - \frac{p_r p_i}{p_r + p_i} \right\} + p_r V_s \right] \qquad \text{(A.6)}$$

or

$$\bar{p} = \frac{1}{V}\left[p_r(V_s + V_t) - \frac{V_t}{3}\left\{2p_r - 2p_i - p_r + \frac{2p_r p_i}{p_r + p_i}\right\}\right] \tag{A.7}$$

Defining $\Delta p = p_r - p_i$ and simplifying Equation (A.7) yields

$$\bar{p} = p_r - 1/3 \frac{V_t}{V} \Delta p \left\{2 - \frac{p_r}{p_r + p_i}\right\} \tag{A.8}$$

Rewriting Equation (A.8) gives

$$p_r - \bar{p} = 1/3 \frac{V_t}{V} \Delta p \left\{1 + \frac{p_i}{p_r + p_i}\right\} \tag{A.9}$$

<u>Case I</u>: Systems wherein $V_s \ll V_t$

In this case $V \simeq V_t$. For systems having line lengths up to approximately 8 feet and line inside diameters greater than 0.125 inch the values of Δp, for present day trajectories, have been found to be of the order of a few tenths of a millimeter of mercury at absolute pressure levels between atmospheric and one millimeter of mercury. Such being the case $p_r \simeq p_i$ and Equation (A.9) can be rewritten as

$$p_r - \bar{p} \simeq 1/2 \, \Delta p \tag{A.10}$$

Thus for pneumatic systems wherein $V_s \ll V_t$ the substitution of $\frac{dp_r}{dt}$ for $\frac{d\bar{p}}{dt}$ in Equation (16) appears justified.

Case II: Systems wherein $V_s \gg V_t$.

Under these conditions it is apparent that the ratio $\frac{V_t}{V} \rightarrow 0$ and Equation (A.9) indicates that $p_r \rightarrow \bar{p}$. Thus for Case II the substitution of $\frac{dp_r}{dt}$ for $\frac{d\bar{p}}{dt}$ would be reasonable.

Case III: Systems wherein $V_s \simeq V_t$.

No definite conclusions can be reached as to the applicability of replacing $\frac{d\bar{p}}{dt}$ by $\frac{dp_r}{dt}$ in Equation (16). However, experimental evidence indicates that the substitution of $\frac{dp_r}{dt}$ for $\frac{d\bar{p}}{dt}$ under the conditions imposed under Case III yields a differential equation which correlates theory with experiment to a very reasonable degree of accuracy.

On the basis of the preceding arguments equation (16) is rewritten as

$$\frac{dp_r}{dt} = \frac{-\pi d^4}{256\mu VL}(p_r^2 - p_i^2) \qquad \text{(A.11)}$$

Equation (A.12) is rewritten as

$$\frac{dp_r}{dt} = K(p_i^2 - p_r^2) \qquad \text{(A.12)}$$

where

$$K = \frac{\pi d^4}{256\mu VL}$$

Equation (A.12) represents the non-linear equation of motion for the flow through the system under the restriction of the assumptions used in its derivation.

One special case in the solution of Equation (A.12) is worthy of note. For the condition of a step input a closed form solution of Equation (A.12) is immediately available as follows: Starting with Equation (A.12) as

$$\frac{dp_r}{dt} = K(p_i^2 - p_r^2)$$

and separating variables gives

$$\int \frac{dp_r}{(p_i^2 - p_r^2)} = \int K\, dt \tag{A.13}$$

The initial conditions are given by

$$\begin{array}{llll} \text{at} & t = 0 & & p_r = p_{r_o} \\ & t > 0 & & p_i = B = \text{constant} \\ & & & p_r = p_r \end{array}$$

The limits on Equation (A.13) may now be imposed and the resulting equation is given by

$$\int_{p_{r_o}}^{p_r} \frac{dp_r}{(B^2 - p_r^2)} = K \int_0^t dt \tag{A.14}$$

or

$$\frac{1}{B} \ln \left(\frac{B + p_r}{B - p_r}\right) \Bigg|_{p_{r_o}}^{p_r} = K\, t \tag{A.15}$$

which becomes

$$\ln\left(\frac{B + p_r}{B - p_r}\right)\left(\frac{B - p_{r_o}}{B + p_{r_o}}\right) = 2B\,Kt \qquad \text{(A.16)}$$

Since $\dfrac{B + p_{r_o}}{B - p_{r_o}}$ is a constant, say, A , Equation (A.16)

can be solved for p_r which is given by

$$p_r = B\,\frac{Ae^{2BKt} - 1}{Ae^{2BKt} + 1} \qquad \text{(A.17)}$$

BIBLIOGRAPHY

1. Huston, W. B., Accuracy of Airspeed Measurements and Flight Calibration Procedures, National Advisory Committee for Aeronautics, NACA TN 1605, June 1948.
2. Wildback, W. A., Pressure Drop in Aircraft Installations, National Advisory Committee for Aeronautics, NACA TN 593, February, 1937.
3. Delio, G. L., Schwent, G. V., and Cesaro, R. S., Transient Behavior of Lumped-Constant Systems for Sensing Gas Pressures, National Advisory Committee for Aeronautics, NACA TN 1988, December, 1949.
4. Vaughn, H., Experimental and Analytical Methods for Determining the Pressure and Time Lag in Pressure-Measuring Systems, Sandia Corporation, SC - 3244 (TR), Albuquerque, New Mexico, May, 1954.
5. Perry, J. A., Critical Flow Through Sharp-Edged Orifices, Transactions of the ASME, October, 1949.
6. Kuethe, A. M., and Schetzer, J. D., Foundations of Aerodynamics, John Wiley and Sons, Inc., New York, 1959, pp. 222-225.